TUTORIAL TEXT No. 1

ELEMENTARY INEQUALITIES

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FOREWORD

The problems contained in this series have been collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time. Hitherto this concentrated material has only been accessible to the very restricted public able to read Serbian*.

I greatly welcome the appearance of the Tutorial Texts based on my problem collection. Cooperation with colleagues in Australia and the U.S.A. and their initiative have made it possible. For the preparation of this Text, I wish to thank E. S. Barnes, D. C. B. Marsh and J. R. M. Radok.

D. S. MITRINOVIĆ.

Belgrade, 23.III.1964.

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* Zbornik matematičkih problema, Belgrade, Vol. I (1962), Vol. II (1960), Vol. III (1960), published in cooperation with D. Adamović, V. Devidé, D. Djoković, D. Mihailović, Z. Pop-Stojanović, S. Prešić, J. Ulčar.

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INTRODUCTION

This tutorial text and problem collection is designed to introduce the student, at undergraduate or senior high school level, to the elementary properties of inequalities. A knowledge of algebra, geometry and trigonometry, together with a first course in calculus, provides a sufficient background for almost all the material. It is hoped that, as a collection of problems, the book will be a useful adjunct to regular course work, while the provision of an introductory survey and numerous complete solutions will give students an opportunity for independent study.

For ease of reference, the material has been roughly classified according to subject matter or method of proof, although a precise classification of inequalities appears to be virtually impossible. For a careful development of the theory of inequalities, the reader is referred to the first two monographs listed below and to the comprehensive bibliographies given there. Many of the problems given in this text are to be found in the problem pages of periodicals such as the *American Mathematical Monthly* which provides an abundant source of both elementary and advanced problems.

G. H. Hardy, J. E. Littlewood, G. Polya: *Inequalities*, Cambridge University Press, London, 1934.

E. F. Beckenbach – R. Bellman: *Inequalities* (Ergebnisse der Mathematik), Julius Springer Verlag, Berlin, 1961.

E. F. Beckenbach – R. Bellman: *An Introduction to Inequalities*, Random House New Mathematical Library, 1961.

N. D. Kazarinoff: *Analytic Inequalities*, Holt-Rinehart and Winston, New York, 1961.

N. D. Kazarinoff: *Geometric Inequalities*, Wesleyan University Press and Random House, 1961.

D. A. Kryzhanovskii: *Elements of the theory of Inequalities*, Moscow-Leningrad, 1936.

G. L. Neviazhskii: *Inequalities*, Moscow, 1947.

P. P. Korovkin: *Inequalities*, Moscow-Leningrad, 1951.

§ 0.1 Inequalities involving Mean Values

0.1.1 *For every set of positive numbers* $A = \{a_1, a_2, \ldots, a_n\}$:

$$\begin{aligned}\min(a_1, a_2, \ldots, a_n) &\leqq \frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}\\ &\leqq (a_1a_2\ldots a_n)^{1/n}\\ &\leqq \frac{a_1+a_2+\ldots+a_n}{n}\\ &\leqq \left(\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{1/2}\\ &\leqq \max(a_1, a_2, \ldots, a_n),\end{aligned}$$

where

$\dfrac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}$ *is the harmonic mean of the numbers* A

$(a_1a_2\ldots a_n)^{1/n}$ *their geometric mean,*

$\dfrac{a_1+a_2+\ldots+a_n}{n}$ *their arithmetic mean and*

$\left(\dfrac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{1/2}$ *their root-mean-square.*

1° Consider first the inequality

$$A_n \equiv \frac{a_1+a_2+\ldots+a_n}{n} \geqq (a_1a_2\ldots a_n)^{1/n} \equiv G_n. \qquad (1$$

PROOF 1. For $n = 2$, this inequality is seen to be true and can be written in the form $(\sqrt{a_1} - \sqrt{a_2})^2 \geqq 0$. Assume that it is valid for $n = k$, i.e.,

$$A_k \geqq G_k.$$

By induction,

$$A \equiv \frac{a_{k+1} + (k-1)A_{k+1}}{k} \geqq (a_{k+1} A_{k+1}^{k-1})^{1/k} \equiv G.$$

Thus, it follows that

$$A_{k+1} = \tfrac{1}{2}(A_k + A) \geqq (A_k A)^{1/2} \geqq (G_k G)^{1/2} = (G_{k+1}^{k+1} A_{k+1}^{k-1})^{1/2k},$$

i.e.,

$$A_{k+1} \geqq (G_{k+1}^{k+1} A_{k+1}^{k-1})^{1/2k},$$

whence

$$A_{k+1} \geqq G_{k+1}.$$

This completes the inductive proof of the inequality (1).

On the basis of this proof, it is easy to show that *equality holds in* (1) *if and only if* $a_1 = a_2 = \ldots = a_n$.

This result is obvious if $n = 2$. Assume that it holds for some $n = k \geqq 2$. From the above proof, we see that if $A_{k+1} = G_{k+1}$, then

$$A_k = A,\ A_k = G_k,\ A = G.$$

Since $A_k = G_k$, we have

$$a_1 = a_2 = \ldots = a_k;$$

and since $A = G$, we have

$$a_{k+1} = A_{k+1} = \frac{a_1 + a_2 + \ldots + a_k + a_{k+1}}{k+1},$$

whence

$$a_1 = a_2 = \ldots = a_k = a_{k+1}.$$

Finally, (1) certainly holds with equality if all a_i are equal. Thus our assertion is established by induction.

PROOF 2. The validity of a statement $P(n)$ can be established by the method of regressive induction in the following manner:

$P(n)$ holds for infinitely many values of n;

$P(n)$ implies $P(n-1)$ for all $n > 1$.

Combine the methods of progressive and regressive induction[1].

It has just been seen that (1) holds for $n = 2$. Assume now that for all positive numbers $a_1, a_2, \ldots, a_n$, some of which may be equal to each other, (1) holds for $n = 2^k$ (where k is some natural number).

Form the sum

$$\frac{a_1+a_2+\ldots+a_\nu}{\nu} \quad (\nu = 2^{k+1} = 2.2^k = 2n),$$

or

$$\frac{1}{2}\left(\frac{a_1+a_2+\ldots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\ldots+a_{2n}}{n}\right).$$

From the inequality

$$\frac{a_1+a_2}{2} \geqq \sqrt{a_1 a_2} \quad (a_1, a_2 > 0), \tag{2}$$

we obtain

$$\frac{1}{2}\left(\frac{a_1+a_2+\ldots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\ldots+a_{2n}}{n}\right)$$

$$\geqq \left(\frac{a_1+a_2+\ldots+a_n}{n} \cdot \frac{a_{n+1}+a_{n+2}+\ldots+a_{2n}}{n}\right)^{\frac{1}{2}}.$$

Using the assumption that the inequality (1) is valid for $n = 2^k$, the last inequality gives

$$\frac{a_1+a_2+\ldots+a_{2n}}{2n} \geqq \{(a_1 a_2 \ldots a_n)^{1/n} (a_{n+1} a_{n+2} \ldots a_{2n})^{1/n}\}^{1/2}$$

$$= (a_1 a_2 \ldots a_{2n})^{1/2n}.$$

Thus, the relation (1) holds for every $n \in \{2, 2^2, 2^3, \ldots\}$. Assume now that the inequality (1) has been proved for n, and replace in it a_n by $(a_1+a_2+\ldots+a_{n-1})/(n-1)$. Then

[1] Alternatively, ascending and descending induction.

$$\frac{a_1+a_2+\ldots+a_{n-1}+\dfrac{a_1+a_2+\ldots+a_{n-1}}{n-1}}{n}$$

$$\geqq \left(a_1 a_2 \ldots a_{n-1}\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right)^{1/n}. \tag{3}$$

When the left-hand side is simplified this relation assumes the form

$$\frac{a_1+a_2+\ldots+a_{n-1}}{n-1} \geqq (a_1 a_2 \ldots a_{n-1})^{1/n}\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right)^{1/n},$$

whence

$$\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right)^{1-1/n} \geqq (a_1 a_2 \ldots a_{n-1})^{1/n}$$

and, finally,

$$\frac{a_1+a_2+\ldots+a_{n-1}}{n-1} \geqq (a_1 a_2 \ldots a_{n-1})^{1/(n-1)}.$$

Thus it follows from the assumption of the truth of the relation (1) for n that it also holds for $n-1$.

This completes the second proof of (1).

2° On the basis of the inequality (1) of 1°, we have for the numbers $1/a_1$, $1/a_2$, . . ., $1/a_n$,

$$\left(\frac{1}{a_1}\frac{1}{a_2}\ldots\frac{1}{a_n}\right)^{1/n} \leqq \frac{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}{n}. \tag{1}$$

This inequality becomes an equality if

$$a_1 = a_2 = \ldots = a_n. \tag{2}$$

It follows from (1) that

$$\frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}} \leqq (a_1 a_2 \ldots a_n)^{1/n}. \tag{3}$$

Consider now the relations:

$$(a_1+a_2+\ldots+a_n)^2 \equiv a_1^2+\ldots+a_n^2 + 2(a_1a_2+a_1a_3+\ldots+a_{n-1}a_n), \quad (4)$$

$$2a_ia_j \leqq a_i^2+a_j^2, \text{ since } (a_i-a_j)^2 \geqq 0. \quad (5)$$

Replacing $2a_ia_j$ by $a_i^2+a_j^2$ on the right-hand side of the identity (4), we obtain the inequality

$$(a_1+a_2+\ldots+a_n)^2 \leqq n(a_1^2+a_2^2+\ldots+a_n^2) \quad (6)$$

which holds for all real a_i. If all the a_i are positive, it follows from (6) that

$$a_1+a_2+\ldots+a_n \leqq \{n(a_1^2+a_2^2+\ldots+a_n^2)\}^{1/2},$$

whence

$$\frac{a_1+a_2+\ldots+a_n}{n} \leqq \left(\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{1/2}. \quad (7)$$

It will now be shown that

$$\min(a_1,a_2,\ldots,a_n) \leqq \frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}. \quad (8)$$

Without loss of generality, suppose that

$$0 < a_1 \leqq a_2 \leqq \ldots \leqq a_n, \quad (9)$$

and hence

$$\min(a_1, a_2, \ldots, a_n) = a_1.$$

Using (9), the inequality (8) becomes

$$\frac{a_1}{a_1}+\frac{a_1}{a_2}+\ldots+\frac{a_1}{a_n} \leqq n.$$

This inequality is valid, because, by (9),

$$a_1/a_k \leqq 1 \ (k = 1, 2, \ldots, n).$$

Consequently, the inequality (8) has been proved.

Next, consider the inequality

$$\left(\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{1/2} \leqq \max(a_1, a_2, \ldots, a_n). \tag{10}$$

Again with the assumption (9), the inequality (10) becomes

$$\left(\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}\right)^{1/2} \leqq a_n,$$

whence

$$a_1^2+a_2^2+\ldots+a_n^2 \leqq na_n^2.$$

This inequality is seen to hold, because, by (9),

$$a_k \leqq a_n \ (k = 1, 2, \ldots, n).$$

Thus, the inequality (10) is valid.

If the a_i are arbitrary real numbers, we have

$$\min(|a_1|, |a_2|, \ldots, |a_n|) \leqq \left(\frac{1}{n}\sum_{k=1}^{n} a_k^2\right)^{1/2} \leqq \max(|a_1|, |a_2|, \ldots, |a_n|).$$

0.1.2 *Prove the inequality*

$$\min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \leqq \frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$$
$$\leqq \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right),$$

where $b_1, b_2, \ldots, b_n$ are positive numbers.

If m is the smallest and M the largest of the numbers

$$a_1/b_1, a_2/b_2, \ldots, a_n/b_n,$$

then

$$m \leqq a_1/b_1 \leqq M, \ m \leqq a_2/b_2 \leqq M, \ldots, m \leqq a_n/b_n \leqq M,$$

whence it follows that

$$m\sum_{k=1}^{n} b_k \leqq \sum_{k=1}^{n} a_k \leqq M\sum_{k=1}^{n} b_k; \text{ i.e., } m \leqq \sum_{k=1}^{n} a_k \Big/ \sum_{k=1}^{n} b_k \leqq M,$$

as was to be proved.

0.1.3 Mean values are important in probability theory, mathematical statistics, and the mathematical analysis of experimental data.

§ 0.2 Bernoulli's Inequality

If $h > -1$ *and* n *is a natural number, then Bernoulli's inequality states that*

$$(1+h)^n \geqq 1+nh. \tag{1}$$

PROOF. Apply the method of mathematical induction. If $n = 1$, the relation (1) holds as an equality. Suppose that the relation (1) holds for $n = k$ ($\geqq 1$), i.e. that

$$(1+h)^k \geqq 1+kh. \tag{2}$$

Multiplying this inequality by $1+h$ (> 0), we obtain

$$(1+h)^{k+1} \geqq (1+h)(1+kh) = 1+(k+1)h+kh^2,$$

whence

$$(1+h)^{k+1} \geqq 1+(k+1)h. \tag{3}$$

Since (2) implies (3), the proof is essentially complete.

Next the generalized Bernoulli inequalities, required in differential calculus, will be established:

$$(1+x)^a > 1+ax \ (-1 < x \neq 0, \ a > 1 \text{ or } a < 0), \tag{4}$$

$$(1+x)^a < 1+ax \ (-1 < x \neq 0, 0 < a < 1). \tag{5}$$

In fact, using Taylor's formula, we obtain

$$(1+x)^a-1-ax = \frac{a(a-1)x^2}{2}(1+\theta x)^{a-2} \quad (0 < \theta < 1). \tag{6}$$

Since, by assumption, $1+\theta x > 0$, this gives

$$\operatorname{sgn}\{(1+x)^a-1-ax\} = \operatorname{sgn}\{a(a-1)\} \quad (x \neq 0),$$

whence the inequalities (4) and (5) follow.

§ 0.3 Chebychev's Inequality

If

$$a_1 \leqq a_2 \leqq \ldots \leqq a_n \text{ and } b_1 \leqq b_2 \leqq \ldots \leqq b_n, \tag{1}$$

then Chebychev's inequality states:

$$\left(\frac{1}{n}\sum_{\nu=1}^{n} a_\nu\right)\left(\frac{1}{n}\sum_{\nu=1}^{n} b_\nu\right) \leqq \frac{1}{n}\sum_{\nu=1}^{n} a_\nu b_\nu . \tag{2}$$

PROOF 1. Let

$$\sum a \equiv \sum_{k=1}^{n} a_k, \quad \sum b \equiv \sum_{k=1}^{n} b_k, \quad \sum ab \equiv \sum_{k=1}^{n} a_k b_k;$$

then

$$\sum_\mu \sum_\nu (a_\mu b_\mu - a_\mu b_\nu) \equiv \sum_\mu (n a_\mu b_\mu - a_\mu \sum b) \equiv n\sum ab - \sum a \sum b,$$

$$\sum_\mu \sum_\nu (a_\nu b_\nu - a_\nu b_\mu) \equiv \sum_\nu (n a_\nu b_\nu - a_\nu \sum b) \equiv n\sum ab - \sum a \sum b.$$

Hence

$$n\sum ab - \sum a \sum b \equiv \tfrac{1}{2}\sum_\mu \sum_\nu (a_\mu b_\mu - a_\mu b_\nu + a_\nu b_\nu - a_\nu b_\mu)$$

$$\equiv \tfrac{1}{2}\sum_\mu \sum_\nu (a_\mu - a_\nu)(b_\mu - b_\nu).$$

By (1),

$$(a_\mu - a_\nu)(b_\mu - b_\nu) \geqq 0 \quad (\mu, \nu = 1, 2, \ldots, n),$$

whence follows Chebychev's inequality,

$$n\sum ab - \sum a \sum b \geqq 0.$$

If and only if

$$a_1 = a_2 = \ldots = a_n \text{ or } b_1 = b_2 = \ldots = b_n,$$

the equality sign holds in (2).

PROOF 2. Let

$$a_1 = p_1, \; a_2 - a_1 = p_2, \ldots, a_n - a_{n-1} = p_n,$$
$$b_1 = q_1, \; b_2 - b_1 = q_2, \ldots, b_n - b_{n-1} = q_n,$$

with $p_\nu \geqq 0$, $q_\nu \geqq 0$ $(\nu = 2, 3, \ldots, n)$, whence (2) may be written as

$$n \sum_{\nu=1}^{n} \left(\sum_{i=1}^{\nu} p_i \right) \left(\sum_{j=1}^{\nu} q_j \right) \geqq \left(\sum_{i=1}^{n} (n+1-i) p_i \right) \left(\sum_{j=1}^{n} (n+1-j) q_j \right)$$

or

$$\sum_{i,j=2}^{n} n[n+1-\max(i, j)] p_i q_j \geqq \sum_{i,j=2}^{n} (n+1-i)(n+1-j) p_i q_j .$$

Since

$$(n+1-i)(n+1-j) \equiv [n+1-\max(i, j)][n+1-\min(i, j)],$$

the last inequality becomes

$$\sum_{i,j=2}^{n} [n+1-\max(i, j)][\min(i, j)-1] p_i q_j \geqq 0. \tag{3}$$

Since

$$n+1-\max(i, j) > 0, \min(i, j)-1 > 0, p_i, q_j \geqq 0$$

for $i, j = 2, 3, \ldots, n$, the inequality (3) is obvious.
This second proof is due to D. Djoković.

GENERALIZATION. If

$$0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n, 0 \leqq b_1 \leqq b_2 \leqq \ldots \leqq b_n, \ldots,$$
$$0 \leqq c_1 \leqq c_2 \leqq \ldots \leqq c_n,$$

then

$$\frac{\sum a}{n} \frac{\sum b}{n} \cdots \frac{\sum c}{n} \leqq \frac{\sum ab \ldots c}{n}.$$

EXAMPLE. If a, b, c, are positive numbers and if n is a natural number, then

$$(a+b+c)^n \leqq 3^{n-1}(a^n+b^n+c^n).$$

REMARKS. *1°* From (3), it follows that equality holds in (2) if and only if

$$a_1 = a_2 = \ldots = a_n \text{ or } b_1 = b_2 = \ldots = b_n.$$

2° In the generalized Chebychev inequality, the conditions $0 \leqq a_1$, $0 \leqq b_1, \ldots, 0 \leqq c_1$ are essential. For instance, if $a_1 = 1$, $a_2 = 3$, $b_1 = 1$, $b_2 = 3$, $c_1 = -4$, $c_2 = -3$, we have

$$\frac{a_1+a_2}{2}\frac{b_1+b_2}{2}\frac{c_1+c_2}{2} = -14 > -31/2 = \frac{a_1b_1c_1+a_2b_2c_2}{2}.$$

In many books (Cf., for instance, J. W. Archbold: *Algebra*, London 1958, pp. 51–52), the conclusions corresponding to 1° and 2° above are incorrect. See also: C. V. Durell-A. Robson: *Advanced Algebra*, vol. III, London 1948, pp. 370–371.

§ 0.4 Abel's Inequality

If $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ $(b_1 \geqq b_2 \geqq \ldots \geqq b_n \geqq 0)$ are two sets of real numbers and if M and m, respectively, are the maximum and minimum of the numbers

$$s_1, s_2, \ldots, s_n \left(s_k = \sum_{\nu=1}^{k} a_\nu\right),$$

then

$$mb_1 \leqq a_1b_1+a_2b_2+\ldots+a_nb_n \leqq Mb_1.$$

Proof. The sum

$$a_1b_1+a_2b_2+\ldots+a_nb_n \left(\equiv \sum_{\nu=1}^{n} a_\nu b_\nu\right)$$

may be written in the form

$$\sum_{\nu=1}^{n} a_\nu b_\nu \equiv s_1b_1+(s_2-s_1)b_2+\ldots+(s_n-s_{n-1})b_n$$
$$\equiv s_1(b_1-b_2)+s_2(b_2-b_3)+\ldots+s_{n-1}(b_{n-1}-b_n)+s_nb_n.$$

Since $M = \max(s_1, s_2, \ldots, s_n)$, i.e.,

$$s_1 \leqq M, s_2 \leqq M, \ldots, s_n \leqq M,$$

and, by hypothesis,

$$b_1-b_2 \geqq 0, b_2-b_3 \geqq 0, \ldots, b_{n-1}-b_n \geqq 0, b_n \geqq 0, \tag{1}$$

we have the sequence of relations

$$s_1(b_1-b_2) \leqq M(b_1-b_2), s_2(b_2-b_3) \leqq M(b_2-b_3), \ldots,$$
$$s_{n-1}(b_{n-1}-b_n) \leqq M(b_{n-1}-b_n), s_nb_n \leqq Mb_n.$$

Adding these expressions we obtain

$$s_1(b_1-b_2)+s_2(b_2-b_3)+\ldots+s_{n-1}(b_{n-1}-b_n)+s_nb_n \leqq Mb_1.$$

Consequently, one arrives at the inequality

$$\sum_{\nu=1}^{n} a_\nu b_\nu \leqq Mb_1. \tag{2}$$

Since $m=\min(s_1, s_2, \ldots, s_n)$, it follows that $m \leqq s_1, \ldots,$ $m \leqq s_n$, and so from (1),

$$m(b_1-b_2) \leqq s_1(b_1-b_2), \ldots, m(b_{n-1}-b_n) \leqq s_{n-1}(b_{n-1}-b_n),$$
$$mb_n \leqq s_nb_n.$$

Hence, summing once again, we obtain

$$mb_1 \leqq \sum_{\nu=1}^{n} a_\nu b_\nu. \tag{3}$$

This result completes the proof of the double inequality

$$mb_1 \leqq \sum_{\nu=1}^{n} a_\nu b_\nu \leqq Mb_1,$$

known as Abel's inequality.

§ 0.5 The Cauchy-Schwarz-Buniakowski Inequality

Consider the two sets of real numbers

$$A=\{a_1, a_2, \ldots, a_n\},\ B=\{b_1, b_2, \ldots, b_n\}$$

and form the polynomial in x:

$$(a_1x+b_1)^2+(a_2x+b_2)^2+\ldots+(a_nx+b_n)^2, \tag{1}$$

which is equal to

$$(a_1^2+a_2^2+\ldots+a_n^2)x^2+2(a_1b_1+a_2b_2+\ldots+a_nb_n)x$$
$$+(b_1^2+b_2^2+\ldots+b_n^2). \tag{2}$$

Since the quadratic trinomial (2) is the sum of squares of real numbers, its value is non-negative for all values of the variable x;

hence

$$(a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2) \\ -(a_1b_1+a_2b_2+\ldots+a_nb_n)^2 \geqq 0, \qquad (3)$$

i.e.,

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \leqq 0, \qquad (4)$$

which is known as the Cauchy-Schwarz-Buniakowski inequality.

Equality holds in (4) if and only if $a_1 = rb_1, a_2 = rb_2, \ldots > a_n = rb_n$, where r is a constant of proportionality.

§ 0.6 Young's Inequality

Let $f(x)$ be a continuous function on the interval $[0, c]$ $(c > 0)$ which is strictly increasing over this interval. Further assume that $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$.

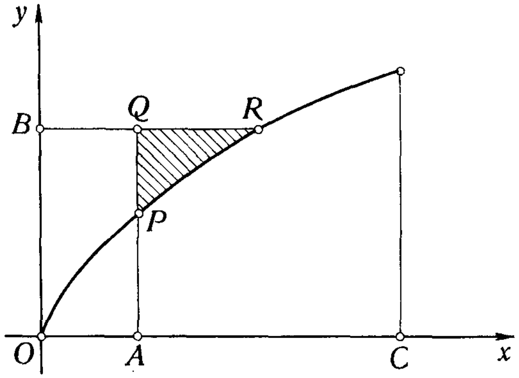

Fig. 1.

The area of the curvilinear triangle OAP (Fig. 1) is given by the integral $\int_0^a f(x)\mathrm{d}x$, while the area of ORB is given by $\int_0^b f^{-1}(x)\mathrm{d}x$, where $f^{-1}(x)$ is the inverse of $f(x)$. On the basis of Figs. 1 and 2, we deduce the inequality

$$\int_0^a f(x)\mathrm{d}x + \int_0^b f^{-1}(x)\mathrm{d}x \geqq ab, \qquad (1)$$

known as Young's inequality.

If and only if $b = f(a)$, equality will hold.

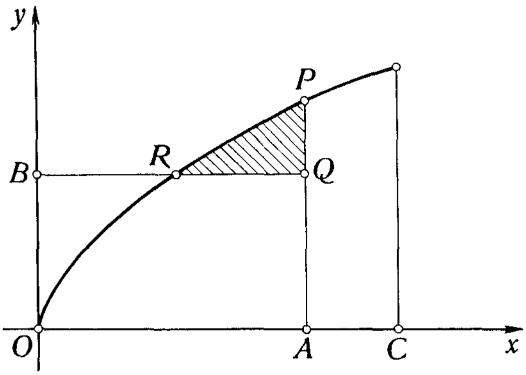

Fig. 2

EXAMPLES. *1°* The function $f(x) \equiv x^{p-1} (p > 1)$ for all x (> 0) satisfies the conditions under which Young's inequality is valid. In this case, the inequality (1) becomes

$$\int_0^a x^{p-1}\,dx + \int_0^b x^{1/(p-1)}\,dx \geqq ab,$$

whence

$$\frac{1}{p} a^p + \frac{p-1}{p} b^{p/(p-1)} \geqq ab.$$

This inequality is usually written in the form

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geqq ab \left(a,\, b \geqq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right). \tag{2}$$

2° The function $\log(1+x)$ also satisfies the conditions of Young's inequality.

In this case, the inequality (1) becomes

$$\int_0^a \log(1+x)\,dx + \int_0^b (e^x - 1)\,dx \geqq ab,$$

whence

$$(1+a)\log(1+a) - (1+a) + (e^b - b) \geqq ab \quad (a,\, b \geqq 0). \tag{3}$$

3° Starting from the inequality (2), we may derive Hölder's inequality

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \geqq \sum_{k=1}^{n} a_k b_k, \tag{4}$$

where

$$a_k,\ b_k \geqq 0,\ p > 1,\ 1/p+1/q = 1.$$

In (2), replace a and b by

$$a = a_\nu \Big/ \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}, \quad b = b_\nu \Big/ \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}$$

to obtain

$$\frac{1}{p}\frac{a_\nu^p}{\sum_{k=1}^{n} a_k^p} + \frac{1}{q}\frac{b_\nu^q}{\sum_{k=1}^{n} b_k^q} \geqq \frac{a_\nu b_\nu}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}}.$$

Summing now over ν from 1 to n we obtain

$$\frac{1}{p} + \frac{1}{q} \geqq \frac{\sum_{k=1}^{n} a_k b_k}{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}},$$

whence Hölder's inequality follows directly, because one may set

$$1/p+1/q = 1.$$

Equality holds in (4) if

$$b_\nu = a_\nu^{p-1} \quad (\nu = 1,2,\ldots, n).$$

4° Hölder's inequality (4) may be used to derive Minkowski's inequality

$$\left(\sum_{k=1}^{n} (a_k+b_k)^p\right)^{1/p} \leqq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \quad (p > 1;\ a_k,\ b_k \geqq 0). \tag{5}$$

Since

$$(a_k+b_k)^p = a_k(a_k+b_k)^{p-1}+b_k(a_k+b_k)^{p-1},$$

we have (writing $\sum$ for $\sum_{k=1}^{n}$)

$$\sum (a_k+b_k)^p = \sum a_k(a_k+b_k)^{p-1} + \sum b_k(a_k+b_k)^{p-1}. \tag{6}$$

We now apply Hölder's inequality to each of the sums on the right-hand side obtaining

$$\sum a_k(a_k+b_k)^{p-1} \leqq \left(\sum a_k^p\right)^{1/p} \left(\sum (a_k+b_k)^{q(p-1)}\right)^{1/q},$$
$$\sum b_k(a_k+b_k)^{p-1} \leqq \left(\sum b_k^p\right)^{1/p} \left(\sum (a_k+b_k)^{q(p-1)}\right)^{1/q}.$$

Inserting these inequalities in (6), and noting that $q(p-1) = p$, we obtain

$$\sum (a_k+b_k)^p \leqq \left[\left(\sum a_k^p\right)^{1/p} + \left(\sum b_k^p\right)^{1/p}\right] \left(\sum (a_k+b_k)^p\right)^{1/q}.$$

Division of this inequality by $\left(\sum (a_k+b_k)^p\right)^{1/q}$ now gives (5).

NOTE. If $p = 1$, (5) is an equality. If, however, $0 < p < 1$, the inequality sign in (5) is reversed.

§ 0.7 Jensen's Inequality

DEFINITION: A function $f(x)$, defined on an interval $[\alpha, \beta]$, is said to be convex on this interval if for every $a, b \in [\alpha, \beta]$

$$f\left(\frac{a+b}{2}\right) \leqq \tfrac{1}{2}\{f(a)+f(b)\}. \tag{1}$$

THEOREM: *For every function $f(x)$, convex on the interval $[\alpha, \beta]$,*

$$f\left(\frac{1}{n}\sum_{\nu=1}^{n} a_\nu\right) \leqq \frac{1}{n}\sum_{\nu=1}^{n} f(a_\nu), \tag{2}$$

where $a_\nu \in [\alpha, \beta]$ and n is a natural number.

PROOF. Assume that the inequality (2) is valid for some natural number $n = 2^k$, i.e., that

$$f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) \leqq \frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n} \quad (n = 2^k). \tag{3}$$

Consider now

$$f\left(\frac{a_1+a_2+\ldots+a_{2n}}{2n}\right) \quad (2n = 2^{k+1}),$$

i.e.,

$$f\left(\frac{\dfrac{a_1+a_2+\ldots+a_n}{n}+\dfrac{a_{n+1}+a_{n+2}+\ldots+a_{2n}}{n}}{2}\right).$$

Using the definition (1) of a convex function, and the relation (3), we obtain

$$f\left(\frac{\dfrac{1}{n}\sum_{\nu=1}^{n} a_\nu+\dfrac{1}{n}\sum_{\nu=1}^{n} a_{\nu+n}}{2}\right) \leqq \frac{f\left(\dfrac{1}{n}\sum_{\nu=1}^{n} a_\nu\right)+f\left(\dfrac{1}{n}\sum_{\nu=1}^{n} a_{\nu+n}\right)}{2}$$

$$\leqq \frac{\sum_{\nu=1}^{n} f(a_\nu)+\sum_{\nu=1}^{n} f(a_{\nu+n})}{2n}$$

$$= \frac{\sum_{\nu=1}^{2n} f(a_\nu)}{2n}.$$

Since the inequality (2) holds for $n = 2^{k+1}$ whenever it holds for $n = 2^k$, and since it holds for $n = 2$ (i.e., for $k = 1$), one arrives at the conclusion that it is valid for every $k \in N$. Thus, the inequality (2) is valid for infinitely many numbers $n \in \{2, 2^2, 2^3, \ldots\}$.

Next it will be shown that the assumption that the inequality (2) is valid for some n implies its validity for $n-1$.

Assume (2) to be valid for some natural number n and for every $a_\nu \in [\alpha, \beta]$. In (2), replace the variable a_n by $(a_1+a_2+\ldots+a_{n-1})/(n-1)$, whence (2) becomes

$$f\left(\frac{a_1+a_2+\ldots+a_{n-1}+(a_1+a_2+\ldots+a_{n-1})/(n-1)}{n}\right)$$

$$\leqq \frac{f(a_1)+f(a_2)+\ldots+f(a_{n-1})+f\left(\dfrac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right)}{n}. \quad (4)$$

The left-hand side of this inequality can now be rewritten as

$$f\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right),$$

whence (4) assumes the form

$$f\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right) \leqq \frac{1}{n}\{f(a_1)+f(a_2)+\ldots+f(a_{n-1})\}$$
$$+\frac{1}{n}f\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right),$$

and it follows that

$$f\left(\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}\right) \leqq \frac{f(a_1)+f(a_2)+\ldots+f(a_{n-1})}{n-1}.$$

Accordingly, if (2) is valid for n, it is also valid for $n-1$. Thus, under the above conditions, the inequality (2) has been proved by the method of regressive induction.

GEOMETRIC INTERPRETATION.

Consider $x_1, x_2, x_3 \in [\alpha, \beta]$, where $x_1 < x_2 < x_3$, and the corresponding functional values $f(x_1), f(x_2), f(x_3)$. The area of the triangle $M_1M_2M_3$ with coordinates $M_1(x_1, f(x_1))$, $M_2(x_2, f(x_2))$, $M_3(x_3, f(x_3))$ is given by $\pm P$, where

$$P = \frac{1}{2}\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix}.$$

The function shown in Fig. 3 is convex, whence $P > 0$, while the function in Fig. 4 is concave and $P < 0$.

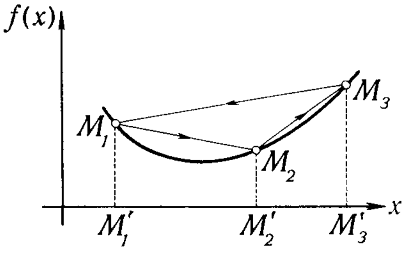

Fig. 3

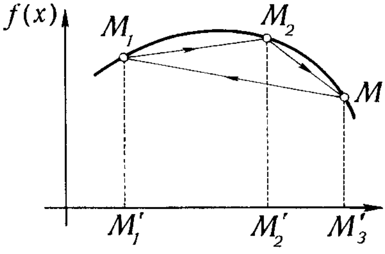

Fig. 4

The condition $P > 0$ corresponds to

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} > 0,$$

i.e.,

$$(x_3-x_2)f(x_1)-(x_3-x_1)f(x_2)+(x_2-x_1)f(x_3) > 0,$$

i.e.,

$$f(x_2) < \frac{x_3-x_2}{x_3-x_1}f(x_1) + \frac{x_2-x_1}{x_3-x_1}f(x_3),$$

since, by assumption, $x_3-x_1 > 0$.

If $x_1 = a$, $x_3 = b$, $x_2 = \frac{1}{2}(a+b)$ $(a, b \in [\alpha, \beta])$, the preceding inequality becomes

$$f\left(\frac{a+b}{2}\right) < \tfrac{1}{2}\{f(a)+f(b)\},$$

because

$$\frac{x_3-x_2}{x_3-x_1} = \frac{x_2-x_1}{x_3-x_1} = \tfrac{1}{2}.$$

In a similar manner, it may be shown that concave functions (Fig. 4) satisfy the inequality

$$f\left(\frac{a+b}{2}\right) \geqq \tfrac{1}{2}\{f(a)+f(b)\}.$$

For a convex function, points on a chord lie above the corresponding points on the arc of the curve $y = f(x)$ (see Fig. 5),

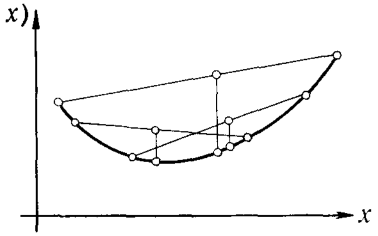

Fig. 5

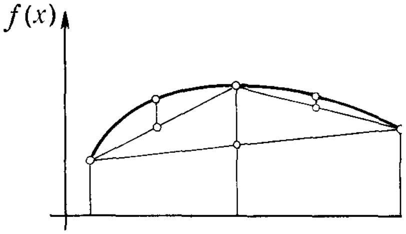

Fig. 6

while for a concave function points on a chord lie below the corresponding points on the arc of the curve (see Fig. 6).

EXAMPLES. *1°* Consider the function $f(x) = x^k$, where k is a natural number > 1, and show, first of all, that it is convex for $x \geqq 0$.

For $k = 2$,

$$f\left(\frac{x_1+x_2}{2}\right) \equiv \frac{x_1^2+x_2^2+2x_1x_2}{4} \quad (x_1, x_2 \in [0, +\infty)).$$

Since $2x_1x_2 \leqq x_1^2+x_2^2$, the preceding inequality implies that

$$f\left(\frac{x_1+x_2}{2}\right) \leqq \frac{x_1^2+x_2^2}{2},$$

i.e.,

$$\left(\frac{x_1+x_2}{2}\right)^2 \leqq \frac{x_1^2+x_2^2}{2}. \tag{1}$$

This shows that the function x^k is convex for $x \geqq 0$ and $k = 2$.

Suppose now that the function $x^k (x \geqq 0)$ is convex for $k = r$ (r any positive integer), i.e.,

$$\left(\frac{x_1+x_2}{2}\right)^r \leqq \frac{x_1^r+x_2^r}{2}. \tag{2}$$

Multiplication by the positive quantity $\frac{1}{2}(x_1+x_2)$ now brings the inequality (2) into the form

$$\left(\frac{x_1+x_2}{2}\right)^{r+1} \leqq \frac{x_1^{r+1}+x_2^{r+1}+x_1^r x_2+x_1x_2^r}{4}. \tag{3}$$

It follows from the identity

$$x_1^r x_2+x_1x_2^r+(x_1-x_2)(x_1^r-x_2^r) \equiv x_1^{r+1}+x_2^{r+1}$$

that

$$x_1^r x_2+x_1x_2^r \leqq x_1^{r+1}+x_2^{r+1},$$

because x_1-x_2 and $x_1^r-x_2^r$ have the same sign.

Introducing this result into (3), we obtain

$$\left(\frac{x_1+x_2}{2}\right)^{r+1} \leqq \frac{x_1^{r+1}+x_2^{r+1}}{2}.$$

This completes the proof that the function x^k (k a natural number > 1, $x \geqq 0$) is convex.

On the basis of our theorem on convex functions, we may write the inequality

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^k \leqq \frac{x_1^k+x_2^k+\ldots+x_n^k}{n} \quad (x_i \geqq 0, i=1, 2, \ldots, n),$$

where equality holds for $x_1 = x_2 = \ldots = x_n$.

2° We will show that on the interval $[0, \pi]$ the function $f(x) \equiv \sin x$ is concave.

For

$$\frac{f(x_1)+f(x_2)}{2} \equiv \frac{\sin x_1 + \sin x_2}{2} \quad (0 \leqq x_1, x_2 \leqq \pi),$$

which may be written in the form

$$\frac{f(x_1)+f(x_2)}{2} \equiv \sin\frac{x_1+x_2}{2}\cos\frac{x_1-x_2}{2};$$

it follows that

$$\frac{f(x_1)+f(x_2)}{2} \leqq \sin\frac{x_1+x_2}{2},$$

because for the range of x_1 and x_2 under consideration

$$0 \leqq \cos\tfrac{1}{2}(x_1-x_2) \leqq 1.$$

Using this fact and the properties of concave functions, we arrive at the relation

$$\frac{1}{n}\sum_{\nu=1}^{n}\sin x_\nu \leqq \sin\left(\frac{1}{n}\sum_{\nu=1}^{n}x_\nu\right) \quad (0 \leqq x_1, x_2, \ldots, x_n \leqq \pi),$$

where equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Sufficient conditions for convexity. The following theorem permits one to determine readily whether a function is convex (concave) over some interval.

Theorem. *If the function $f(x)$ has a second derivative $f''(x)$ over the interval $[\alpha, \beta]$ which satisfies the inequality*

$$f''(x) \geqq 0 \quad (x \in [\alpha, \beta]), \tag{1}$$

then the function $f(x)$ is convex over the interval $[\alpha, \beta]$.

PROOF. Let x_1 and x_2 be any two points on the interval $[\alpha, \beta]$. Applying Taylor's formula in the neighbourhood of the point $\frac{1}{2}(x_1+x_2)$, we obtain

$$f(x_1) = f\left(\frac{x_1+x_2}{2}\right) + \left(x_1 - \frac{x_1+x_2}{2}\right) f'\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}\left(x_1 - \frac{x_1+x_2}{2}\right)^2 f''(\xi) \quad \left\{\xi \in \left(x_1, \frac{x_1+x_2}{2}\right)\right\}, \tag{2}$$

$$f(x_2) = f\left(\frac{x_1+x_2}{2}\right) + \left(x_2 - \frac{x_1+x_2}{2}\right) f'\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}\left(x_2 - \frac{x_1+x_2}{2}\right)^2 f''(\eta) \quad \left\{\eta \in \left(\frac{x_1+x_2}{2}, x_2\right)\right\}. \tag{3}$$

From (2) and (3), it follows that

$$\frac{f(x_1)+f(x_2)}{2} = f\left(\frac{x_1+x_2}{2}\right) + \tfrac{1}{16}(x_2-x_1)^2[f''(\xi)+f''(\eta)]. \tag{4}$$

It follows from (1) that $f''(\xi) \geqq 0$ and $f''(\eta) \geqq 0$, whence we conclude from (4) that

$$\frac{f(x_1)+f(x_2)}{2} \geqq f\left(\frac{x_1+x_2}{2}\right) \qquad (\alpha \leqq x_1, x_2 \leqq \beta),$$

i.e., that the function $f(x)$ is convex on the interval $[\alpha, \beta]$, as was to be proved.

Since $f(x)$ is concave if and only if $-f(x)$ is convex, we deduce that, if $f''(x) \leqq 0$ over the interval $[\alpha, \beta]$, then the function $f(x)$ is concave over this interval.

EXAMPLES. *1°* Since $(x^k)'' = k(k-1)x^{k-2} \geqq 0$ for $k \geqq 1$ and $x \geqq 0$, the function $x^k (k \geqq 1)$ is convex for $x \geqq 0$.

2° Since $(\sin x)'' = -\sin x \leqq 0$ for $x \in [0, \pi]$, the function $\sin x$ is concave over the interval $[0, \pi]$.

3° Since $(\log x)'' = -1/x^2 < 0$ for $x > 0$, $\log x$ is concave on $(0, \infty)$. It follows therefore that, if $a_k > 0$ $(k = 1, \ldots, n)$,

$$\log\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) \geqq \frac{1}{n}\sum_{k=1}^{n} \log a_k = \log\,(a_1 a_2 \ldots a_n)^{1/n}.$$

Since log x is increasing, this gives

$$A_n = \frac{a_1+a_2+\ldots+a_n}{n} \geqq (a_1 a_2 \ldots a_n)^{1/n} = G_n.$$

This is an alternative proof of the inequality of the arithmetic and geometric means (§ 0.1).

§ 0.8 The Fejér-Jackson Inequality

L. Vietoris [1] has proved the inequalities:

$$\sum_{k=1}^{n} a_k \sin kx > 0, \quad \sum_{k=0}^{n} a_k \cos kx > 0 \quad (0 < x < \pi), \tag{1}$$

provided that

$$a_0 \geqq a_1 \geqq \ldots \geqq a_n > 0, \tag{2}$$

$$a_{2k} \leqq \frac{2k-1}{2k} a_{2k-1} \quad (1 \leqq k \leqq \tfrac{1}{2}n). \tag{3}$$

If $a_0 = 1$ and $a_k = 1/k$ ($k = 1, 2, \ldots, n$), the conditions (2) and (3) are fulfilled and the inequalities (1) become

$$\sum_{k=1}^{n} \frac{1}{k} \sin kx > 0 \quad (0 < x < \pi), \tag{4}$$

$$1 + \sum_{k=1}^{n} \frac{1}{k} \cos kx > 0 \quad (0 < x < \pi). \tag{5}$$

The inequality (4) is known as the Fejér-Jackson inequality.

§ 0.9 Jordan's Inequality

Since $\sec^2 \theta \geqq 1$ for $0 \leqq \theta < \frac{1}{2}\pi$, we have, integrating over $[0, \theta]$,

$$\tan \theta \geqq \theta \text{ for } 0 \leqq \theta < \tfrac{1}{2}\pi.$$

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[1] Über das Vorzeichen gewisser trigonometrischer Summen (*Sitz. Ber. Öst. Ak. Wiss.*, Bd. **167**, 1958, S. 125—135; – *Anzeiger Öst. Ak. Wiss.*, 1959, S. 192—193).

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\theta}\right) = \frac{\cos\theta}{\theta^2}(\theta - \tan\theta) \leqq 0 \text{ for } 0 < \theta < \tfrac{1}{2}\pi.$$

Thus, $\sin\theta/\theta$ is continuous and decreasing on $(0, \frac{1}{2}\pi]$, whence

$$\frac{\sin\theta}{\theta} \geqq \frac{2}{\pi} \text{ for } 0 < \theta \leqq \tfrac{1}{2}\pi.$$

This inequality is known as Jordan's inequality. Equality holds only when $\theta = \frac{1}{2}\pi$.

Combining this with the inequality

$$\sin\theta \leqq \theta \quad (\theta \geqq 0),$$

we deduce that

$$\frac{2}{\pi} \leqq \frac{\sin\theta}{\theta} \leqq 1 \text{ for } |\theta| \leqq \tfrac{1}{2}\pi.$$

§ 0.10 Some Integral Inequalities

The inequalities of Cauchy-Schwarz-Buniakowski and Hölder (§§ 0.5, 0.6) have integral analogues, which may be proved by similar arguments. All functions occurring here are assumed to be integrable over an interval $[\alpha, \beta]$.

Schwarz's Inequality

$$\left|\int_\alpha^\beta f(x)g(x)\,\mathrm{d}x\right|^2 \leqq \int_\alpha^\beta |f(x)|^2\,\mathrm{d}x \int_\alpha^\beta |g(x)|^2\,\mathrm{d}x. \tag{1}$$

Proof. For all real t,

$$(tf(x)+g(x))^2 \geqq 0,$$

and so

$$\int_\alpha^\beta (tf(x)+g(x))^2\,\mathrm{d}x \geqq 0,$$

$$t^2\int_\alpha^\beta f^2(x)\,\mathrm{d}x + 2t\int_\alpha^\beta f(x)g(x)\,\mathrm{d}x + \int_\alpha^\beta g^2(x)\,\mathrm{d}x \geqq 0, \text{ all } t.$$

This quadratic in t therefore has a non-negative discriminant, from which (1) follows immediately.

HÖLDER'S INEQUALITY

$$\left|\int_\alpha^\beta f(x)g(x)\mathrm{d}x\right| \leqq \left\{\int_\alpha^\beta |f(x)|^p \mathrm{d}x\right\}^{1/p} \left\{\int_\alpha^\beta |g(x)|^q \mathrm{d}x\right\}^{1/q}$$

$$\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right). \qquad (2)$$

PROOF. We again use the inequality (2) of § 0.6, namely,

$$ab \leqq \frac{1}{p}\, a^p + \frac{1}{q}\, b^q \quad \left(a, b \geqq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1\right). \qquad (3)$$

Set

$$A = \left(\int_\alpha^\beta |f(x)|^p \mathrm{d}x\right)^{1/p}, \quad B = \left(\int_\alpha^\beta |g(x)|^q \mathrm{d}x\right)^{1/q}$$

and replace a, b in (3) by $|f(x)|/A$, $|g(x)|/B$, respectively. We obtain

$$\frac{1}{AB}|f(x)g(x)| \leqq \frac{1}{pA^p}|f(x)|^p + \frac{1}{qB^q}|g(x)|^q,$$

whence by integration

$$\frac{1}{AB}\int_\alpha^\beta |f(x)g(x)|\,\mathrm{d}x \leqq \frac{1}{pA^p}\cdot A^p + \frac{1}{qB^q}\cdot B^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus

$$\left|\int_\alpha^\beta f(x)g(x)\,\mathrm{d}x\right| \leqq \int_\alpha^\beta |f(x)g(x)|\,\mathrm{d}x \leqq AB,$$

and this is Hölder's Inequality (2).

§ 0.11 Some Inequalities for symmetric functions [1]

1. Consider the function

$$f(x) = (x-a)^p(x-b)^q(x-c)^r,$$

where a, b, c, p $(\geqq 1)$, $q(\geqq 1)$, $r(\geqq 1)$ are real constants and $a \neq b \neq c \neq a$, and its first derivative

[1] D. S. Mitrinović: O nekim nejednakostima, *Publikacije Elektrotehničkog fakulteta u Beogradu*, ser. Matematika i fizika, No. 29 (1959), 1–4.

$$f'(x) = (x-a)^{p-1}(x-b)^{q-1}(x-c)^{r-1}\{(p+q+r)x^2 - [p(b+c)+q(c+a)+r(a+b)]x+(pbc+qca+rab)\}.$$

Without loss of generality, it may be assumed that $a < b < c$.

The function $f(x)$ has the three zeros $x = a$, $x = b$, $x = c$. By Rolle's Theorem, the derivative has a zero between a and b:

$$\alpha = \frac{1}{2(p+q+r)}\{p(b+c)+q(c+a)+r(a+b) - [(p(b+c)+q(c+a)+r(a+b))^2 - 4(p+q+r)(pbc+qca+rab)]^{1/2}\}.$$

Another zero occurs between b and c:

$$\beta = \frac{1}{2(p+q+r)}\{p(b+c)+q(c+a)+r(a+b) + [(p(b+c)+q(c+a)+r(a+b))^2 - 4(p+q+r)(pbc+qca+rab)]^{1/2}\}.$$

Since these zeros of the derivative, which lie in the intervals (a, b) and (b, c), are real and distinct, we have

$$[p(b+c)+q(c+a)+r(a+b)]^2-4(p+q+r)(pbc+qca+rab) > 0.$$

We also obtain the inequality

$$\min(a, b, c) < \alpha < \operatorname{med}(a, b, c) < \beta < \max(a, b, c),$$

where $\operatorname{med}(a, b, c)$ denotes that one of the numbers a, b, c which lies between $\max(a, b, c)$ and $\min(a, b, c)$.

2. Consider the function

$$g(x) = (x-a)(x-b)(x-c)(x-d) = x^4-(\sum a)x^3+(\sum ab)x^2-(\sum abc)x+abcd,$$

where a, b, c, d are distinct real numbers, and its first derivative

$$g'(x) = 4x^3-3(\sum a)x^2+2(\sum ab)x-\sum abc,$$

where $\sum a$, $\sum ab$, $\sum abc$ are the elementary symmetric functions of the variables a, b, c, d of degrees 1, 2, 3, respectively. All three zeros of the third order polynomial $g'(x)$ are real and distinct. The condition that all three zeros of a polynomial

$$a_0x^3+3a_1x^2+3a_2x+a_3$$

are real and distinct is $G^2+4H^3 < 0$, where

$$H = a_0a_2-a_1^2, \quad G = a_0^2a_3-3a_0a_1a_2+2a_1^3.$$

For the above function H and G are found to be

$$H = \tfrac{8}{3}\sum ab-(\sum a)^2,$$
$$G = -16\sum abc+8(\sum a)(\sum ab)-2(\sum a)^3.$$

As a result we find the inequality

$$27[8\sum abc-4(\sum a)(\sum ab)+(\sum a)^3]^2+[8\sum ab-3(\sum a)^2]^3 < 0,$$

or

$$108(\sum abc)^2-9(\sum ab)^2(\sum a)^2-108\sum a\sum ab\sum abc$$
$$+27(\sum a)^3\sum abc+32(\sum ab)^3 < 0.$$

Hence we arrive at the following inequalities:

1° If $\sum a = 0$, then $27(\sum abc)^2+8(\sum ab)^3 < 0$.

2° If $\sum ab = 0$, then $4(\sum abc)^2+(\sum a)^3\sum abc < 0$.

3° If $\sum abc = 0$, then $32\sum ab-9(\sum a)^2 < 0$.

3. Applying this procedure for forming inequalities to the function

$$h(x) = (x-a)^p(x-b)^q(x-c)^r(x-d)^s,$$

where a, b, c, d are real, distinct numbers, $p \geqq 1$, $q \geqq 1$, $r \geqq 1$ $s \geqq 1$, one finds an inequality which contains the preceding inequality as a special case.

The first derivative of the function $h(x)$ is

$$h'(x) = (x-a)^{p-1}(x-b)^{q-1}(x-c)^{r-1}(x-d)^{s-1}$$
$$\times \{p(x-b)(x-c)(x-d)+q(x-c)(x-d)(x-a)$$
$$+r(x-d)(x-a)(x-b)+s(x-a)(x-b)(x-c)\},$$

or

$$h'(x) = (x-a)^{p-1}(x-b)^{q-1}(x-c)^{r-1}(x-d)^{s-1}$$
$$\times \{(p+q+r+s)x^3-[p(b+c+d)+q(c+d+a)+r(d+a+b)$$
$$+s(a+b+c)]x^2+[p(cd+db+bc)+q(da+ac+cd)$$
$$+r(ab+bd+da)+s(bc+ca+ab)]x$$
$$-(pbcd+qcda+rdab+sabc)\}.$$

The third order polynomial $P(x) = a_0x^3+3a_1x^2+3a_2x+a_3$ contained within the braces has three real, distinct zeros, and hence

$$G^2+4H^3 < 0, \tag{1}$$

where $H = a_0a_2-a_1^2$, $G = a_0^2a_3-3a_0a_1a_2+2a_1^3$.

It follows from this that

$$\begin{aligned} a_0 &= p+q+r+s, \\ -3a_1 &= p(b+c+d)+q(c+d+a)+r(d+a+b)+s(a+b+c), \\ 3a_2 &= p(cd+db+bc)+q(da+ac+cd)+r(ab+bd+da) \\ &\qquad +s(bc+ca+ab), \\ -a_3 &= pbcd+qcda+rdab+sabc. \end{aligned}$$

For instance, if

$$p(b+c+d)+q(c+d+a)+r(d+a+b)+s(a+b+c) = 0,$$

the condition (1) becomes

$$4a_0^3a_2^3+a_0^4a_3^2 < 0,$$

whence

$$4a_2^3+a_0a_3^2 < 0, \text{ because } a_0 = p+q+r+s > 0,$$

and

$$4[p(cd+db+bc)+q(da+ac+cd)+r(ab+bd+da)+s(bc+ca+ab)]^3$$
$$-27(p+q+r+s)(pbcd+qcda+rdab+sabc)^2 < 0.$$

If the zeros of the polynomial $P(x)$ are denoted by x_1, x_2, x_3 ($x_1 < x_2 < x_3$) and we assume that

$$a < b < c < d,$$

we have the inequalities

$$a < x_1 < b < x_2 < c < x_3 < d,$$

where x_1, x_2, x_3 are functions of the parameters p, q, r, s, a, b, c, d which may be determined by Cardan's formulae.

§ 1. Elementary Inequalities

1.1 Prove the inequality

$$(n!)^2 < k!(2n-k)!,$$

where n and k ($< n$) are natural numbers.

PROOF. If the given inequality is divided by $n!k!$, we obtain

$$n(n-1) \ldots (k+1) < (n+1)(n+2) \ldots (2n-k).$$

This inequality is obvious, for the products on the left- and right-hand sides contain the same number of factors and each factor on the left-hand side is less than each factor on the right-hand side.

1.2 Prove that

$$f(x) = 2x^3+3x^2-12x+7 > 0,$$

if $x > 1$. For which values of x is $f(x) < 0$?

HINT. Express $f(x)$ as a polynomial in $(x-1)$.

1.3 Prove the inequality

$$(2k)! < 2^{2k}(k!)^2 \quad (k \text{ a natural number}). \tag{1}$$

METHOD 1. For $k = 1$, this inequality is valid. Consider any k for which (1) holds. If the inequality

$$(2k+1)(2k+2) < 2^2(k+1)^2 \quad (k \geqq 1) \tag{2}$$

is true, it follows from (1) and (2) that

$$(2k+2)! < 2^{2k+2}\{(k+1)!\}^2. \tag{3}$$

The validity of (2) may be established without difficulty, for it is equivalent to the inequality

$$0 < 2k+2$$

which holds for $k > -1$. Thus an inductive proof is easily constructed.

METHOD 2. Since[1]

$$(2k)! = (2k)!!(2k-1)!!, \quad 2^{2k}(k!)^2 = \{(2k)!!\}^2,$$

the given inequality may be written in the form

$$\frac{(2k)!!}{(2k-1)!!} = \prod_{\nu=1}^{k} \frac{2\nu}{2\nu-1} > 1.$$

This last inequality is obvious, because

$$\frac{2\nu}{2\nu-1} > 1 \quad (\nu = 1, 2, \ldots, k).$$

1.4 Prove the inequality

$$|a+b|^p \leqq 2^p\,(|a|^p+|b|^p) \quad (p \geqq 1).$$

PROOF. Without loss of generality, suppose that $|a| \leqq |b|$. Then

$$|a+b| \leqq 2|b|,$$

whence

$$|a+b|^p \leqq 2^p|b|^p \leqq 2^p(|a|^p+|b|^p).$$

1.5 If $a > b > 0$, prove the inequality

$$\sqrt[n]{a}-\sqrt[n]{b} < \sqrt[n]{a-b} \quad (n \text{ a natural number} > 1). \tag{1}$$

SOLUTION 1. Setting $a-b = c > 0$, inequality (1) may be rewritten

$$\sqrt[n]{b+c} < \sqrt[n]{b}+\sqrt[n]{c}. \tag{2}$$

Assume that the inequality (2) does not hold for some b, c (b, $c > 0$), i.e., that

$$\sqrt[n]{b+c} \geqq \sqrt[n]{b}+\sqrt[n]{c}. \tag{3}$$

Then

$$(\sqrt[n]{b+c})^n \geqq (\sqrt[n]{b}+\sqrt[n]{c})^n,$$

whence

[1] $n!! = n(n-2)(n-4)\ldots$, where the last factor is 1 or 2.

$$b+c \geqq b+c+\sum_{k=1}^{n-1}\binom{n}{k}(\sqrt[n]{b})^{n-k}(\sqrt[n]{c})^{k}$$

and

$$0 \geqq \sum_{k=1}^{n-1}\binom{n}{k}(\sqrt[n]{b})^{n-k}(\sqrt[n]{c})^{k}. \tag{4}$$

Thus, assuming (3) to be true, we are led to the false relation (4). Accordingly, the inequality (1) holds subject to the stated assumptions regarding the parameters n, a, b.
(This solution is due to D. Djoković.)

SOLUTION 2. Consider the function

$$f(x) = x^{1/n}-(x-1)^{1/n} \quad (n > 1, x \geqq 1).$$

Then

$$nf'(x) = x^{1/n-1}-(x-1)^{1/n-1} < 0 \quad (x > 1).$$

Therefore $f(x)$ decreases for $x > 1$. Hence, since $f(1) = 1$, we have $f(x) < 1$ for $x > 1$, i.e.,

$$x^{1/n}-1 < (x-1)^{1/n} \quad (x > 1). \tag{5}$$

The inequality (5) is satisfied for $x = a/b > 1$, because $a > b > 0$. Therefore, setting $x = a/b$, we have

$$\left(\frac{a}{b}\right)^{1/n}-1 < \left(\frac{a}{b}-1\right)^{1/n},$$

i.e.,

$$a^{1/n}-b^{1/n} < (a-b)^{1/n},$$

as was to be proved.

1.6 If $a > b > 0$, then

$$\sqrt{a^2+k^2}-\sqrt{b^2+k^2} \leqq a-b. \tag{1}$$

Prove this inequality and find a bound for

$$\sqrt[n]{a^n+k^n}-\sqrt[n]{b^n+k^n} \quad (n \text{ a natural number}, k \geqq 0, a > b > 0).$$

Moreover, how must the inequality (1) be modified when a and b are arbitrary real numbers?

SOLUTION. Start with the identity

$$(x-y)(x^{n-1}+x^{n-2}y+\ldots+xy^{n-2}+y^{n-1}) = x^n-y^n \qquad (2)$$

and set

$$x = \sqrt[n]{a^n+k^n},\ y = \sqrt[n]{b^n+k^n}.$$

Since $x \geqq a$ and $y \geqq b$ ($k \geqq 0$) and $a > b > 0$, we obtain from (2)

$$(\sqrt[n]{a^n+k^n}-\sqrt[n]{b^n+k^n})(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}) \leqq a^n-b^n.$$

If the left- and right-hand members of the last inequality are divided by the positive number

$$a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1},$$

we obtain

$$\sqrt[n]{a^n+k^n}-\sqrt[n]{b^n+k^n} \leqq a-b. \qquad (3)$$

For $n = 2$, the inequality (3) becomes

$$\sqrt{a^2+k^2}-\sqrt{b^2+k^2} \leqq a-b.$$

The inequality holds for $a \geqq b \geqq 0$, while k may be an arbitrary real number (positive, negative or zero).

If a, b, k are any arbitrary real numbers, then

$$\left|\sqrt{a^2+k^2}-\sqrt{b^2+k^2}\right| \leqq \big||a|-|b|\big|.$$

(This solution is due to D. Djoković.)

1.7 If a, b, c, d are real numbers and if

$$ad-bc = 1, \qquad (1)$$

prove that $E \equiv a^2+b^2+c^2+d^2+ac+bd > 1$.

SOLUTION. The expression E may be written in the form

$$E \equiv 1+\tfrac{1}{2}[(a+c)^2+(a-d)^2+(b+d)^2+(b+c)^2].$$

The expression in brackets vanishes if and only if

$$a+c = 0,\ a-d = 0,\ b+d = 0,\ b+c = 0, \qquad (2)$$

i.e., if $a = b = c = d = 0$. But if this holds, (1) is not satisfied.

Therefore $E > 1$ for all real a, b, c, d satisfying (1). (See **1.23**).

1.8 Solve the pair of inequalities

$$\frac{2x-y}{y} < 0, \quad \frac{2y-x}{x} < 0 \qquad (x, y \neq 0).$$

SOLUTION. Setting $x/y = z$, the given inequalities become

$$2z-1 < 0, \quad \frac{2}{z}-1 < 0.$$

It follows from the first of these inequalities that

$$z < \tfrac{1}{2}, \tag{1}$$

and from the second that

$$z < 0 \text{ or } z > 2. \tag{2}$$

It follows from (1) and (2) that the given inequalities are both satisfied only if $z < 0$, i.e., if x and y have opposite signs.

1.9 Solve graphically the inequality

$$\big||x|-1\big| \leqq a \ (a \geqq 0).$$

1.10 Show that for all a

$$f(a) = \cos 3a+4\cos 2a+8\cos a+5 \geqq 0.$$

SOLUTION. $f(a) = 4\cos^3 a+8\cos^2 a+5\cos a+1$
$= (2\cos a+1)^2(\cos a+1) \Rightarrow (\forall a) f(a) \geqq 0.$

1.11 Prove the inequality

$$x^m y^n+x^n y^m < x^{m+n}+y^{m+n} \quad (x \neq y;\ x, y > 0;\ m, n > 0).$$

HINT. Consider the expression

$$x^{m+n}-x^m y^n-x^n y^m+y^{m+n}, \text{ i.e., } (x^m-y^m)(x^n-y^n).$$

1.12 Prove the inequality

$$x^{2n-1}+x < x^{2n}+1 \,(x \neq 1;\ x > 0;\ n \text{ a natural number}),$$

and hence the inequality

$$x^{n-1}+\frac{1}{x^{n-1}} < x^n+\frac{1}{x^n} \quad (x \neq 1;\, x > 0;\ n \text{ a natural number}).$$

HINT. Consider the product $(x^{2n-1}-1)(x-1)$. — See the preceding problem.

1.13 Prove that if $a^2+b^2 = c^2+d^2 = 1$, then

$$|ac+bd| \leqq 1.$$

HINT. Write

$$a = \cos x, \quad b = \sin x, \quad c = \cos y, \quad d = \sin y.$$

1.14 Given the function

$$f(x) = x+\log x, \tag{1}$$

find the values of x and h for which the inequality

$$F(x, h) = f(x+h)+f(x-h)-2f(x) < 0 \tag{2}$$

holds.

SOLUTION. The function $f(x)$ is defined for $x > 0$, and for $x > 0$, $x+h > 0$, $x-h > 0$,

$$F(x, h) = \log(x+h)+\log(x-h)-2\log x.$$

Since

$$F(x, h) = \log\frac{x^2-h^2}{x^2},$$

the inequality (2) holds for $0 < |h| < x$.

1.15 Prove the inequalities

$$f(2x) > f(x) > f(-x) > f(-2x) \qquad (x > 0),$$
$$f(2x)-f(-2x) > 2\{f(x)-f(-x)\} \qquad (x > 0),$$

where $f(x) = (a^x-1)/x$ $(a > 1)$.

What happens if $x < 0$?

1.16 Prove the inequality

$$\sqrt{a^2-b^2}+\sqrt{2ab-b^2} > a \quad (0 < b < a). \tag{1}$$

HINT. Assume that the relation (1) does not hold, i.e., that for some a, b $(0 < b < a)$

$$\sqrt{a^2-b^2}+\sqrt{2ab-b^2} \leqq a.$$

1.17 Prove the inequality

$$a+b-(2-\sqrt{2})\sqrt{ab} \leqq \sqrt{a^2+b^2} \quad (a,\, b \geqq 0). \tag{1}$$

SOLUTION. Suppose that (1) does not hold, but that for some a, b $(a, b \geqq 0)$

$$a+b-(2-\sqrt{2})\sqrt{ab} > \sqrt{a^2+b^2} \quad (\geqq 0). \tag{2}$$

Taking the square of (2) leads to

$$2ab-(a+b)\sqrt{ab} > 0$$

which implies (since, from (2), $a \neq 0$ and $b \neq 0$)

$$4a^2b^2 > (a+b)^2ab \Leftrightarrow 4ab > (a+b)^2,$$

i.e.,

$$0 > (a-b)^2. \tag{3}$$

This relation does not hold for any values of a, b, Hence it follows that (1) is always true.

1.18 Prove the inequality

$$1+\cot a \leqq \cot \frac{a}{2} \quad (0 < a < \pi).$$

SOLUTION. $\cot \frac{a}{2} - \cot a = \operatorname{cosec} a \geqq 1.$

1.19 Let

$$|x-a|+|y-b| < \varepsilon \quad (a,\, b \text{ real};\ \varepsilon > 0).$$

Does it then follow that

$$|xy-ab| < (|a|+|b|+\varepsilon)\varepsilon?$$

1.20 For what values of x is

$$|ax+b| < c \quad (a \neq 0,\, c > 0)?$$

ANSWER.

$$\frac{c-b}{a} > x > -\frac{c+b}{a} \quad (a > 0),$$

$$-\frac{c+b}{a} > x > \frac{c-b}{a} \quad (a < 0).$$

1.21 For what values of x is

$$\frac{a|x|+1}{x} < 1?$$

ANSWER.

$$\frac{1}{1+a} < x < 0 \text{ or } x > \frac{1}{1-a} \quad (a < -1),$$

$$x < 0 \text{ or } x > \tfrac{1}{2} \quad (a = -1),$$

$$x < 0 \text{ or } x > \frac{1}{1-a} \quad (-1 < a < +1),$$

$$x < 0 \quad (a \geqq 1).$$

1.22 Let

$$f(a, b, c, d) = (a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2.$$

If $a < b < c < d$, prove that

$$f(a, c, b, d) > f(a, b, c, d) > f(a, b, d, c).$$

HINT. Begin by considering the differences

$$f(a, b, c, d)-f(a, b, d, c),$$
$$f(a, c, b, d)-f(a, b, c, d).$$

Generalise to the case

$$f(a, b, c, d, e) = (a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-a)^2,$$

where $a < b < c < d < e$.

Also treat the case

$$f(a_1, a_2, \ldots, a_n) = (a_1-a_2)^2+(a_2-a_3)^2 + \ldots +(a_{n-1}-a_n)^2+(a_n-a_1)^2.$$

1.23 Prove the inequality

$$S = a^2+b^2+c^2+d^2+ac+bd \geqq \sqrt{3}, \tag{1}$$

where a, b, c, d are real and $ad-bc = 1$.

SOLUTION. The inequality (1) follows immediately from the identity

$$S-\sqrt{3}= \left(a+\frac{c}{2}-\frac{d}{2}\sqrt{3}\right)^2+\left(b+\frac{c}{2}\sqrt{3}+\frac{d}{2}\right)^2.$$

In (1), equality holds if

$$a^2 = \tfrac{1}{2}\sqrt{3}, \quad b = -\frac{a}{3}\sqrt{3}, \quad c = 0 \quad \text{and} \quad d = \frac{2a}{3}\sqrt{3}.$$

1.24 Find a natural number N such that

$$\sum_{n=1}^{\infty} \frac{1}{n+n^3} - \sum_{n=1}^{N} \frac{1}{n+n^3} < 0.02. \tag{1}$$

SOLUTION. We must choose N so that

$$R_N = \sum_{n=N+1}^{\infty} \frac{1}{n+n^3} < 0.02.$$

A Cauchy approximation of the sum of the series $\sum_{n=1}^{\infty} 1/(n+n^3)$ by an integral gives

$$R_N < \int_N^{+\infty} \frac{\mathrm{d}x}{x+x^3} = \int_N^{+\infty} \left(\frac{1}{x}-\frac{x}{x^2+1}\right)\mathrm{d}x$$

$$= \log\left[\frac{x}{\sqrt{x^2+1}}\right]_N^{+\infty}$$

$$= \tfrac{1}{2}\log\left(1+\frac{1}{N^2}\right).$$

Since $\log(1+x) < x$ $(x > 0)$, we have

$$R_N < \tfrac{1}{2}\log\left(1+\frac{1}{N^2}\right) < \frac{1}{2N^2}.$$

Hence (1) holds if N satisfies the inequality

$$\frac{1}{2N^2} \leqq \frac{2}{100} \Rightarrow N \geqq 5.$$

Thus, $N = 5$ suffices.

1.25 Prove or disprove the following inequalities:

1°: $[a]+[b] \leqq [a+b] \leqq [a]+[b]+1$,

2°: $[a][b] \leqq [ab] \leqq [a][b]+[a]+[b]$,

3°: $[\sqrt{a}] = [\sqrt{[a]}]$,

4°: $[\sqrt{n}]^2 \leqq n \leqq [\sqrt{n}]^2+2[\sqrt{n}]$,

5°: $[\sqrt[3]{n}]^3 \leqq n \leqq [\sqrt[3]{n}]^3+3[\sqrt[3]{n}]^2+3[\sqrt[3]{n}]$.

(In 1° and 2°, a, b are real numbers, in 3°, $a \geqq 0$, in 4°, $n =$ 0, 1, 2, . . ., in 5°, n is integral).

1.26 If a, b, t are positive numbers, show that $at+b/t \geqq 2\sqrt{ab}$.

1.27 By graphical methods, estimate the solutions of the inequality

$$\sqrt{x}+\sqrt{x-1} > \sqrt{x+1}.$$

1.28 Show that the number

$$M = \frac{a+kb}{1+k} \quad (a,\, b \text{ real};\ k > 0)$$

lies between a and b.

1.29 Determine the region of the xy-plane in which the point (x, y) must lie in order that its coordinates satisfy the inequality

$$(x^2-4xy)/(x^2+3xy+2y^2) < 0.$$

1.30 Find the region of the plane in a Cartesian coordinate system whose points (x, y) satisfy the condition

$$\big||x+a|-|y-a|\big| < a \quad (a > 0).$$

HINT. Possible cases are

$$\begin{aligned} &1°: x+a > 0, y-a > 0,\\ &2°: x+a > 0, y-a < 0,\\ &3°: x+a < 0, y-a > 0,\\ &4°: x+a < 0, y-a < 0, \end{aligned} \tag{1}$$

corresponding to the set of inequalities

$$\begin{aligned} &1°: -a < x-y+2a < a,\\ &2°: -a < x+y < a,\\ &3°: -a < x+y < a,\\ &4°: -a < x-y+2a < a. \end{aligned} \tag{2}$$

Is the desired region just the interior of the square in Fig. 7?

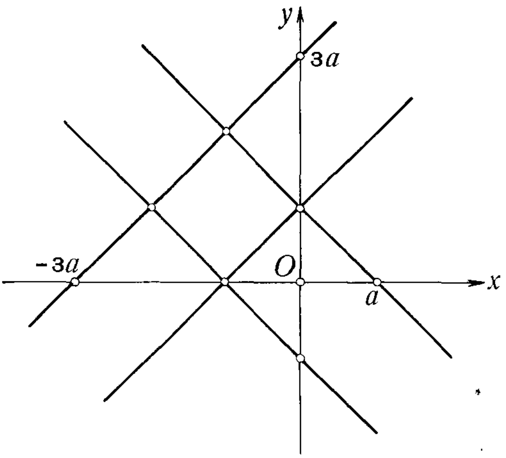

Fig. 7

1.31 Determine graphically the solutions of $|x+1|+|y-2| \leqq 1$.

1.32 Show that if (1) $a \geqq 1$, (2) $b+c < a+1$, (3) $b \leqq c$, then (4) $b < a$.

PROOF 1. Suppose that (4) is false, so that

$$a \leqq b. \tag{5}$$

From (2) and (5) it follows that

$$b+c < b+1$$

or

$$c < 1. \qquad (6)$$

From (3) and (5) we find that $a \leqq c$ and, by (6),

$$a \leqq c < 1 \Rightarrow a < 1,$$

which contradicts (1). Thus, the inequality (4) holds under the conditions (1), (2), (3).

PROOF 2. By the condition (3), one has also

$$2b \leqq b+c.$$

By (2) and the preceding inequality, one finds

$$2b < a+1.$$

By (1) and this inequality

$$2b < 2a \Rightarrow b < a,$$

which is the desired result.

1.33 Show that if x is real, the function $4x(1-x)/(1+x)^2$ cannot assume values greater than 1/2.

1.34 For what value or values of a is the condition

$$(x^2+ax+1)/(x^2+4x+8) < 8$$

satisfied for all real x?

1.35 Determine k such that for all real x

$$\left|(x^2-kx+1)/(x^2+x+1)\right| < 3.$$

HINT. The given inequality is equivalent to

$$-3 < (x^2-kx+1)/(x^2+x+1) < +3.$$

RESULT. $k \in (-5, +1)$.

1.36 Solve the inequality $\{(3x-1)/(2-x)\}^{1/2} > 1$.

RESULT. $x \in (3/4, 2)$.

1.37 For what values of x is it true that

$$(1-\sqrt{1-8x^2})/(2x) \leqq 1?$$

1.38 For what values of x is it true that

$$2 < (3x^2-15x+16)/(x^2-4x+3) < 3?$$

1.39 In a Cartesian coordinate system, find the region of the plane for which

$$1°: \quad xy(x^2-y^2) > 0, \qquad 2°: \quad (x^2-1)(x^2-y^2) < 0.$$

1.40 In a Cartesian coordinate system, find the region of the plane

$$\{(x, y) : y < x\} \cap \{(x, y) : y > -\tfrac{1}{2}(x-3)\}$$

i.e., find the region whose points (x, y) satisfy the conditions

$$y < x, \ y > -\tfrac{1}{2}(x-3).$$

Similarly, find the following regions

$$\{(x, y) : y^2 < x\} \cap \{(x, y) : x^2+y^2 > 1\},$$
$$\{(x, y) : y^3 < x\} \cap \{(x, y) : x < y^2\}.$$

1.41 Determine the region of the xy-plane containing those points (x, y) whose coordinates satisfy $\sin(x+y) > 0$.

1.42 Solve the inequality $(\sin 3x)/(\sin x)^3 < 0$.

1.43 Solve the inequality $(\tan 3x)/(\tan x) > 0$.

1.44 Solve the inequality $\cos \phi+\sin \phi > 1$ (e.g., by setting $x = \cos \phi$, $y = \sin \phi$).

1.45 For which values of x is the inequality

$$\sin x > 2 \cos^2 x-1 \text{ valid?}$$

HINT. Establish this first for $x \in [0, T]$, where T is the fundamental period of the function $\sin x - 2\cos^2 x + 1$.

1.46 Solve the inequalities:

1°: $x+|x| < 1$, 2°: $x-|x| > 2$,

3°: $|x^2-x|+x > 1$, 4°: $\sin x+|\sin x| > 1$.

1.47 Solve the simultaneous inequalities:

$$y^2+4x-4 > 0, \quad 8x^2-2x-3y-6 > 0.$$

1.48 If $a+b+c = 0$ and $a \geqq -\frac{1}{4}$, $b \geqq -\frac{1}{4}$, $c \geqq -\frac{1}{4}$, prove that

$$(4a+1)^{\frac{1}{2}}+(4b+1)^{\frac{1}{2}}+(4c+1)^{\frac{1}{2}} \leqq 3.$$

HINT. $(4a+1)^{\frac{1}{2}} \leqq 2a+1$ $(a \geqq -\frac{1}{4})$. Generalize this result.

1.49 Find the region of the xy-plane for which the following inequalities are simultaneously satisfied:

$$x^2 < 7y,\ y^2 > 5x,\ y^2 < 8x,\ x^2 > 2y.$$

1.50 Find the region of the xy-plane for which the following inequalities are simultaneously satisfied:

$$xy \leqq b,\ xy \geqq a,\ y \leqq mx,\ y \geqq kx\ (0 < a < b;\ 0 < k < m).$$

1.51 For which values of a does the following inequality hold:

$$-1 < \frac{1}{2a}\left[1-a-\sqrt{(1-a)^2-4a^2}\right] < +1?$$

RESULT. $-1 < a < 1/3$ $(a \neq 0)$.

REMARK. May these results be obtained by considering the quadratic equation whose roots are

$$\frac{1}{2a}\left[1-a\pm\sqrt{(1-a)^2-4a^2}\right]?$$

1.52 If a and b $(ab \neq 0)$ are arbitrary real numbers, prove that at least one of the following inequalities is valid:

$$\left|\frac{a+\sqrt{a^2+2b^2}}{2b}\right| < 1, \left|\frac{a-\sqrt{a^2+2b^2}}{2b}\right| < 1.$$

PROOF. Since

$$\left|\frac{a+\sqrt{a^2+2b^2}}{2b}\right| \cdot \left|\frac{a-\sqrt{a^2+2b^2}}{2b}\right| = \tfrac{1}{2},$$

then at least one of these factors must be less than 1.

1.53 Prove the implication

$$\left|\frac{x^2-2x+3}{x^2-4x+3}\right| \leqq 1 \Rightarrow x \leqq 0.$$

1.54 Prove the inequality

$$\frac{p+m}{p-m} \geqq \frac{x^2-2mx+p^2}{x^2+2mx+p^2} \geqq \frac{p-m}{p+m} \quad (p > m > 0). \tag{1}$$

PROOF. First of all, we have

$$y = \frac{x^2-2mx+p^2}{x^2+2mx+p^2} = -1+\frac{2(x^2+p^2)}{x^2+2mx+p^2}$$
$$= -1+\frac{2}{1+\dfrac{2mx}{x^2+p^2}}. \tag{2}$$

From (2), we see that y assumes smaller values for $x > 0$ than for $x < 0$. Thus it suffices, when bounding y from below, to examine only the case $x > 0$.

For $x > 0$, we may write in succession

$$x^2+p^2 \geqq 2px, \frac{1}{x^2+p^2} \leqq \frac{1}{2px}, \frac{2mx}{x^2+p^2} \leqq \frac{m}{p},$$

$$1+\frac{2mx}{x^2+p^2} \leqq 1+\frac{m}{p} = \frac{m+p}{p}, \quad \frac{2}{1+\dfrac{2mx}{x^2+p^2}} \geqq \frac{2p}{p+m},$$

$$y \geqq -1+\frac{2p}{p+m} = \frac{p-m}{p+m}. \tag{3}$$

For $x < 0$, we obtain, similarly,

$$x^2+p^2 \geqq -2px, \frac{1}{x^2+p^2} \leqq -\frac{1}{2px}, \frac{2mx}{x^2+p^2} \geqq -\frac{m}{p},$$

$$1+\frac{2mx}{x^2+p^2} \geqq 1-\frac{m}{p} = \frac{p-m}{p}, \frac{2}{1+\dfrac{2mx}{x^2+p^2}} \leqq \frac{2p}{p-m},$$

$$y \leqq -1+\frac{2p}{p-m} = \frac{p+m}{p-m}. \tag{4}$$

Thus, (3) and (4) imply (1).

1.55 If $0 < a < 2c$, determine whether $(x+a)/(x^2+ax+c^2)$ lies between $-(2c+a)^{-1}$ and $(2c-a)^{-1}$.

1.56 If $a = \cos\alpha$, $c = \sin\alpha$, $b^2 = \sin 2\alpha$ $(0 < \alpha < \pi/4)$, and

$$f(x) = (ax^2+bx+c)/(cx^2+bx+a),$$

prove that

$$(\sec\alpha-1)(\operatorname{cosec}\alpha+1) \leqq f(x) \leqq (\sec\alpha+1)(\operatorname{cosec}\alpha-1).$$

1.57 Find lower and upper bounds for the function

$$(x^2-2x\cos a+1)/(x^2-2x\cos b+1).$$

1.58 Determine pairs of integers, x and y, which satisfy simultaneously

$$y-|x^2-2x|+\tfrac{1}{2} > 0,\ y+|x-1| < 2.$$

SOLUTION. We write the inequalities in the form

$$y+\tfrac{1}{2} > |x^2-2x|,\ 2-y > |x-1|. \tag{1}$$

Since the moduli $|x^2-2x|$ and $|x-1|$ are non-negative, it follows from (1) that $y > -\frac{1}{2}$ and $y < 2$; these inequalities yield the integral values $y = 0$ or $y = 1$.

If we substitute $y = 0$ in (1), we have the simultaneous inequalities

$$|x^2-2x| < \tfrac{1}{2},\ |x-1| < 2.$$

Integral solutions of the first inequality are $x = 0, 2$ and of the second, $x = 0, 1, 2$. Consequently, the common solutions are $x = 0, 2$. Thus, pairs of integral solutions corresponding to $y = 0$ are:

$$x_1 = 0, y_1 = 0; \quad x_2 = 2, y_2 = 0.$$

Substituting $y = 1$ in (1), we obtain

$$|x^2-2x| < \tfrac{3}{2}, \quad |x-1| < 1.$$

The integral solutions of the first inequality are $x = 0, 1, 2$ and of the second just $x = 1$. Hence the third integral solution is

$$x_3 = 1, \quad y_3 = 1.$$

1.59 $a^2+b^2+c^2 \geqq |bc+ca+ab|$.

1.60 $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \geqq 0$ $(a, b, c \geqq 0)$ (Schur).

1.61 $\dfrac{ab}{a+b}+\dfrac{cd}{c+d} \leqq \dfrac{(a+c)(b+d)}{a+b+c+d}$ $(a, b, c, d > 0)$.

1.62 $1+a+a^2+\ldots+a^n < \dfrac{1}{1-a}$ $(0 < a < 1)$.

1.63 $\sqrt{a+b} \leqq \sqrt{a}+\sqrt{b}$ $(a, b \geqq 0)$.

1.64 $a+b-\tfrac{2}{3}\sqrt{ab} \leqq \sqrt{a^2+b^2} \leqq a+b$ $(a, b \geqq 0)$.

1.65 $a+\dfrac{b}{2a} > \sqrt{a^2+b} > a+\dfrac{b}{2a}-\dfrac{1}{2a}\left(\dfrac{b}{2a}\right)^2$ $(a, b > 0)$.

1.66 $\{a-(n+1)(a-b)\}a^n < b^{n+1}$ $(a \neq b;\ ab > 0)$.

1.67 $2^n > n$ $(n \geqq 1)$.

1.68 $2^n > 2n+1$ $(n \geqq 3)$.

1.69 $2^n > n^2$ $(n \geqq 5)$.

1.70 $2^n > n^3$ $(n \geqq 10)$.

1.71 $3^n > n^4$ $(n \geqq 8)$.

1.72 $|x^n-a^n| \leqq nr^{n-1}|x-a|$ $(|x| \leqq r;\ |a| \leqq r)$.

1.73 $\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n} \quad (n > 1).$

1.74 $\left(1+\frac{1}{6n}\right)^{-n} > \frac{5}{6}.$

1.75 $(1-a)^n \geqq 1-na \quad (0 \leqq a < 1).$

1.76 $(1-a)^n < \frac{1}{1+na} \quad (0 < a < 1).$

1.77 $(1+a)^n < \frac{1}{1-na} \quad \left(0 < a < \frac{1}{n}\right).$

1.78 $\sqrt[n]{1+a} > \frac{1}{1-\frac{a}{n(1+a)}} > 1+\frac{a}{n(1+a)} \quad (a > 0;\ n > 1).$

§ 2. Inequalities Obtainable from Functional Considerations

2.1 Prove the inequality

$$\log(1+x) < \frac{x(x+2)}{2(x+1)} \quad (x > 0). \tag{1}$$

SOLUTION. Let

$$f(x) = \log(1+x) - \frac{x(x+2)}{2(x+1)}.$$

Since $f'(x) = -x^2/2(1+x)^2 < 0$, $f(x) < f(0) = 0 \quad (x > 0)$, which is precisely the inequality (1).

2.2 Prove the inequality

$$1+\frac{x}{5} \geqq (1+x)^{1/5} \quad (x \geqq 0).$$

Does this inequality hold for negative values of x?

2.3 Let $f(t) = t-\frac{1}{6}t^3+\frac{1}{24}t^4 \sin\frac{1}{t} \quad (t > 0).$

Prove that, if $x > 0, z > 0, x+z < 1$,

$$f(x+z) < f(x)+f(z).$$

SOLUTION. Let $g(t) = f(t)/t$. For $0 < t < 1$,

$$g'(t) = -\frac{t}{24}\left\{8-3t\sin\frac{1}{t}+\cos\frac{1}{t}\right\} < 0.$$

Note first that $g(t)$ is decreasing in the interval (0,1). It follows from $x > 0$, $z > 0$, $x+z < 1$ that $g(x+z) < g(x)$ and $g(x+z) < g(z)$. Therefore

$$xg(x+z)+zg(x+z) < xg(x)+zg(z).$$

Using the definition of $g(t)$, we obtain from the last formula

$$f(x+z) < f(x)+f(z).$$

(Solution due to D.C.B. Marsh).

2.4 Prove the inequality

$$|a+b|^p \leqq |a|^p+|b|^p \quad (0 \leqq p \leqq 1). \tag{1}$$

PROOF. If a and b have opposite signs the result is immediately evident. Otherwise, let $t = b/a$ $(a \neq 0)$; the relation (1) becomes

$$(1+t)^p \leqq 1+t^p \quad (0 \leqq p \leqq 1).$$

For $p = 0$ or $p = 1$, this result is trivial. Consider the function

$$(1+t)^p-1-t^p \quad (0 < p < 1)$$

which vanishes for $t = 0$ and decreases as t increases. This yields the inequality (1) for $a \neq 0$. (1) also holds when $a = 0$.

What modification should be made if $p > 1$?

2.5 Given the function

$$h(x) = \begin{cases} f(x)/g(x) & (0 < |x| < 2\pi), \\ \lim\limits_{x\to 0} f(x)/g(x) & (x = 0), \end{cases}$$

where

$$f(x) = 2-2\cos x - x\sin x \text{ and } g(x) = x^2-\sin^2 x;$$

prove the inequality

$$h(x) > 0 \quad (-2\pi < x < 2\pi). \tag{1}$$

SOLUTION. The value of the above limit is 1/4.

The function $h(x)$ is even and thus may be considered only for $0 < x < 2\pi$.

First of all, we have

$$g(x) = x^2-\sin^2 x > 0 \quad (0 < x < 2\pi).$$

Consider $f(x)$ written in the form

$$f(x) = 4\sin^2(\tfrac{1}{2}x) - 4(\tfrac{1}{2}x)\sin(\tfrac{1}{2}x)\cos(\tfrac{1}{2}x)$$
$$= 4\sin^2(\tfrac{1}{2}x)\left\{1-\frac{\frac{1}{2}x}{\tan(\frac{1}{2}x)}\right\} \quad (x \neq \pi).$$

If $0 < x < \pi$, $\tfrac{1}{2}x < \tan(\tfrac{1}{2}x)$; consequently,

$$1-\frac{\frac{1}{2}x}{\tan(\frac{1}{2}x)} > 0 \quad (0 < x < \pi).$$

Then, for $0 < x < \pi$, $f(x) > 0$.

If $\pi < x < 2\pi$, $\tan(\tfrac{1}{2}x) < 0$ and, consequently,

$$1-\frac{\frac{1}{2}x}{\tan(\frac{1}{2}x)} > 0 \quad (\pi < x < 2\pi).$$

Thus, for $\pi < x < 2\pi$, $f(x) > 0$.

When $x = \pi$, one has

$$f(\pi) = [2-2\cos x - x\sin x]_{x=\pi} = 4 > 0.$$

This proves the inequality (1).

2.6 Prove the inequality

$$y\log\frac{y}{x} + (1-y)\log\frac{1-y}{1-x} \geqq 2(y-x)^2 \quad (0 < x, y < 1). \tag{1}$$

PROOF. Consider the function

$$f(x) = y\log\frac{y}{x} + (1-y)\log\frac{1-y}{1-x} - 2(y-x)^2 \quad (0 < x, y < 1),$$

where y is a parameter. Then

$$f'(x) = -\frac{y}{x}+\frac{1-y}{1-x}+4(y-x) = (x-y)\frac{(2x-1)^2}{x(1-x)}.$$

Since $0 < x < 1$, we find

$$f'(x) \geqq 0 \quad (x \geqq y),$$
$$f'(x) \leqq 0 \quad (x \leqq y),$$

whence it follows that $\min f(x) = f(y) = 0$.

This proves the inequality (1).

2.7 Prove the inequality

$$px^q - qx^p > p-q \quad (x > 1), \tag{1}$$

where p and q are real numbers $(0 < p < q)$.

SOLUTION. Consider the function

$$f(x) = px^q - qx^p - p + q$$

and its derivative

$$f'(x) = pq(x^{q-1}-x^{p-1}).$$

Taking into consideration the conditions on p and q, we see that $f'(x)$ is positive for $x > 1$, i.e., the function is increasing on the interval $(1, +\infty)$. The function $f(x)$ therefore attains a minimum at $x = 1$ which implies that $f_{\min} = 0$. Thus, $f(x) > 0$ for $x > 1$, which was to be proved.

2.8 Prove the inequality

$$(y-x)a^x \log a < a^y - a^x < (y-x)a^y \log a \quad (x < y, a > 1). \tag{1}$$

SOLUTION. The function $f(t) = a^t (a > 1)$ satisfies the conditions for the Mean Value Theorem on the interval $[x, y]$. Hence

$$\frac{a^y - a^x}{y-x} = a^c \log a \quad (x < c < y). \tag{2}$$

Since the function $f(t)$ $(a > 1)$ is increasing, it follows from (2) that

$$a^x \log a < \frac{a^y - a^x}{y - x} < a^y \log a \quad (x < y;\ a > 1),$$

which is equivalent to (1). This completes the proof.

2.9 Prove the inequality

$$1 - x \geqq e^{-x-x^2} \quad (|x| \text{ sufficiently small}).$$

SOLUTION. We set $f(x) = e^{-x-x^2}$ and determine the equation of the tangent to the curve $y = f(x)$ at the point $(0,1)$.

Since $f'(x) = -(1+2x)e^{-x-x^2}$, the equation of this tangent is

$$y = 1 - x.$$

Since

$$f''(x) = (4x^2 + 4x - 1)e^{-x-x^2} < 0$$

for

$$x \in \left(-\tfrac{1}{2}(1+\sqrt{2}),\ \tfrac{1}{2}(\sqrt{2}-1)\right),$$

we conclude that $y = f(x)$ is convex upwards in the neighbourhood of $(0,1)$.

Consequently,

$$1 - x \geqq e^{-x-x^2} \text{ for } |x| \text{ sufficiently small.}$$

2.10 Prove the inequality

$$0 \leqq \frac{x \log x}{x^2 - 1} \leqq \tfrac{1}{2} \quad (x > 0,\ x \neq 1). \tag{1}$$

Using this result, prove that the integral

$$\int_0^1 \frac{x^m \log x}{x^2 - 1}\, dx = \int_0^1 x^{m-1} \frac{x \log x}{x^2 - 1}\, dx \quad (m \geqq 1)$$

is not greater than $1/(2m)$.

SOLUTION. Consider the function

$$f(x) = 2 \log x - \frac{x^2 - 1}{x}.$$

Since

$$f'(x) = \frac{2}{x} - 1 - \frac{1}{x^2} = -\left(\frac{x-1}{x}\right)^2 \leqq 0,$$

the function $f(x)$ is decreasing and, consequently,

$$2 \log x - \frac{x^2-1}{x} < 0 \quad (x > 1),$$

$$2 \log x - \frac{x^2-1}{x} > 0 \quad (0 < x < 1).$$

Hence it follows that

$$\frac{x \log x}{x^2-1} \leqq \tfrac{1}{2}.$$

The inequality

$$0 \leqq \frac{x \log x}{x^2-1}$$

is obvious, because for $x > 0$,

$$\operatorname{sgn}(x \log x) = \operatorname{sgn}(x^2-1).$$

It follows from (1) that

$$\int_0^1 \frac{x^m \log x}{x^2-1} \, dx \leqq \tfrac{1}{2} \int_0^1 x^{m-1} \, dx = 1/(2m).$$

(Revised from a solution by Z. Pop-Stojanović.)

2.11 For which values of x is

$$\log x \leqq \sqrt{x}?$$

2.12 Sketch the graph of the function

$$\sqrt{x} - \sqrt{x-a} \quad (a > 0)$$

and find the values of x for which

$$\sqrt{x} - \sqrt{x-a} > 2.$$

2.13 Find the values of x for which

$$\sqrt[n]{x}-\sqrt[n]{x-a} > b \quad (n \text{ a natural number} > 1;\ a, b > 0).$$

2.14 Factor the polynomial $2x^3-3x^2+1$ and determine the values of x for which it is positive. Hence prove the inequality

$$f(x) = 2\cos x+\sec^2 x-3 > 0 \quad (0 < x < \tfrac{1}{2}\pi).$$

By considering the function

$$\int_0^x f(t)\,dt \quad (0 < x < \tfrac{1}{2}\pi),$$

show that

$$2\sin x+\tan x > 3x \quad (0 < x < \tfrac{1}{2}\pi).$$

SOLUTION. First of all, we have

$$2x^3-3x^2+1 = (x-1)^2(2x+1).$$

Hence

$$2x^3-3x^2+1 > 0 \quad (x > -\tfrac{1}{2} \text{ and } x \neq 1), \tag{1}$$

$$2x^3-3x^2+1 < 0 \quad (x < -\tfrac{1}{2}). \tag{2}$$

Since $0 < \cos\theta < 1$ for $0 < \theta < \frac{1}{2}\pi$, we find from (1)

$$2\cos^3\theta-3\cos^2\theta+1 > 0 \quad (0 < \theta < \tfrac{1}{2}\pi),$$

or

$$2\cos\theta+\sec^2\theta-3 > 0 \qquad (0 < \theta < \tfrac{1}{2}\pi). \tag{3}$$

Integrating the inequality (3) between the limits $(0, x)$, we obtain

$$2\sin x+\tan x-3x > 0 \quad (0 < x < \tfrac{1}{2}\pi),$$

which was to be proved.

2.15 Prove

$$a < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} < b \quad (0 < a < b). \tag{1}$$

SOLUTION. Applying the Theorem of the Mean to the function $x\log x$, we obtain

$$b \log b - a \log a = (b-a)(1+\log c) \quad (0 < a < c < b),$$

i.e.,

$$\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} = c,$$

whence (1) follows immediately.

2.16 Compare the magnitudes of the functions

$$(\sqrt{n})^{\sqrt{n+1}} \text{ and } (\sqrt{n+1})^{\sqrt{n}} \quad (n \text{ a natural number}).$$

SOLUTION. Since

$$(\sqrt{n})^{\sqrt{n+1}} > (\sqrt{n+1})^{\sqrt{n}} \Leftrightarrow \frac{\log \sqrt{n}}{\sqrt{n}} > \frac{\log \sqrt{n+1}}{\sqrt{n+1}}, \tag{1}$$

$$(\sqrt{n})^{\sqrt{n+1}} < (\sqrt{n+1})^{\sqrt{n}} \Leftrightarrow \frac{\log \sqrt{n}}{\sqrt{n}} < \frac{\log \sqrt{n+1}}{\sqrt{n+1}}, \tag{2}$$

we compare the functions

$$\frac{\log \sqrt{n}}{\sqrt{n}} \text{ and } \frac{\log \sqrt{n+1}}{\sqrt{n+1}}.$$

Without difficulty, we establish that the function $f(x) = (\log x)/x$ is increasing for $0 < x < e$ and decreasing for $x > e$. It follows from this result that

$$\frac{\log \sqrt{n}}{\sqrt{n}} < \frac{\log \sqrt{n+1}}{\sqrt{n+1}} \quad (n = 1, 2, 3, 4, 5, 6), \tag{3}$$

$$\frac{\log \sqrt{n}}{\sqrt{n}} > \frac{\log \sqrt{n+1}}{\sqrt{n+1}} \quad (n > 7), \tag{4}$$

since $\sqrt{7} < e < \sqrt{8}$. For $n = 7$, direct calculations show that the inequality (4) holds.

From (1), (2), (3), (4), we have the inequalities

$$(\sqrt{n})^{\sqrt{n+1}} < (\sqrt{n+1})^{\sqrt{n}} \quad (n = 1, 2, 3, 4, 5, 6),$$

$$(\sqrt{n})^{\sqrt{n+1}} > (\sqrt{n+1})^{\sqrt{n}} \quad (n \geqq 7).$$

2.17 Prove that $x-\tanh x > 0$ $(x > 0)$, and hence that

$$\frac{d}{dx}\left(\frac{\sinh x}{x}\right) > 0 \quad (x > 0).$$

SOLUTION. Consider the function

$$f(x) = x-\tanh x. \tag{1}$$

Since

$$f'(x) = \tanh^2 x > 0 \quad (x \neq 0), \tag{2}$$

the function $f(x)$ increases for $x > 0$. It follows from $f(0) = 0$ and (2) that

$$f(x) = x-\tanh x > 0 \quad (x > 0). \tag{3}$$

By (3), it follows from the relation

$$\frac{d}{dx}\left(\frac{\sinh x}{x}\right) = \frac{\cosh x}{x^2}(x-\tanh x)$$

and the inequality $\cosh x > 0$ that

$$\frac{d}{dx}\left(\frac{\sinh x}{x}\right) > 0 \quad (x > 0).$$

2.18 Prove the inequality

$$\sqrt[4]{x} \leqq 2x+\tfrac{3}{8} \quad (x \geqq 0). \tag{1}$$

SOLUTION. Consider the function

$$f(x) = x^{1/4}-2x \quad (x \geqq 0)$$

and its derivatives:

$$f'(x) = \tfrac{1}{4}x^{-3/4}-2, \quad f''(x) = -\tfrac{3}{16}x^{-7/4} \quad (x > 0).$$

For $x = \frac{1}{16}$, the derivative $f'(x)$ vanishes, while $f''(x) < 0$. Therefore $f(x)$ attains its maximum value of $\frac{3}{8}$ at $x = \frac{1}{16}$. Consequently,

$$\sqrt[4]{x}-2x \leqq \tfrac{3}{8} \quad (x > 0),$$

which was to be proved.

For $x = 0$, the inequality obviously holds.

2.19 Prove the inequality

$$t(2+\cos t) > 3\sin t \quad (t > 0). \tag{1}$$

SOLUTION. Since $2+\cos t > 0$ for all t, we may write (1) in the form

$$f(t) = t - \frac{3\sin t}{2+\cos t} > 0.$$

Since

$$f'(t) = \left(\frac{1-\cos t}{2+\cos t}\right)^2 \geqq 0,$$

the function $f(t)$ increases as t increases. Since $f(0) = 0$, this establishes the inequality (1).

2.20 For $x(\neq 1)$ an arbitrary positive number, prove that

1°: $x^p-1 > p(x-1) \quad (p > 1 \text{ or } p < 0)$,
2°: $x^p-1 < p(x-1) \quad (0 < p < 1)$.

PROOF. 1°: Consider the function

$f(x) = x^p-1-p(x-1)$, for which $f(1) = 0$, $f'(x) = p(x^{p-1}-1)$.

If $p > 1$, then

$$f'(x) = \begin{cases} < 0 & (0 < x < 1), \\ = 0 & (x = 1), \\ > 0 & (x > 1). \end{cases}$$

Thus, the function $f(x)$ attains a minimum at $x = 1$.

This is the proof of the inequality 1° for $p > 1$.

If $p < 0$, then

$$x^{p-1} > 1 \quad (0 < x < 1),$$
$$x^{p-1} < 1 \quad (x > 1).$$

Hence

$$f'(x) \begin{cases} < 0 & (0 < x < 1), \\ = 0 & (x = 1), \\ > 0 & (x > 1). \end{cases}$$

Consequently, 1° also holds for $p < 0$.

2°: If $0 < p < 1$,

$$x^{p-1} > 1 \quad (0 < x < 1),$$
$$x^{p-1} < 1 \quad (x > 1).$$

Hence

$$f'(x) \begin{cases} > 0 & (0 < x < 1), \\ = 0 & (x = 1), \\ < 0 & (x > 1). \end{cases}$$

The function $f(x)$ attains a maximum at $x = 1$.

This proves the inequality 2°.

2.21 If a and b ($|a| \leqq 1$, $|b| \leqq 1$) are real numbers, then

$$\sqrt{1-a^2}+\sqrt{1-b^2} \leqq 2\sqrt{1-\{\tfrac{1}{2}(a+b)\}^2}. \tag{1}$$

SOLUTION. Assume that (1) does not hold, but that

$$\sqrt{1-a^2}+\sqrt{1-b^2} > 2\sqrt{1-\tfrac{1}{4}(a+b)^2}.$$

Then, squaring both sides, we find

$$\sqrt{1-a^2}\,\sqrt{1-b^2} > 1-ab,$$

i.e., squaring again,

$$(1-a^2)(1-b^2) > (1-ab)^2 \Rightarrow 0 > (a-b)^2.$$

On the basis of this contradiction, we conclude that the inequality (1) is true.

2.22 If $f'(x)$ is an increasing function, then

$$f'(x+1) > f(x+1)-f(x) > f'(x). \tag{1}$$

Use this result to prove the inequality

$$\tfrac{2}{3}\{\sqrt{(n+1)^3}-1\} > \sum_{k=1}^{n} \sqrt{k} > \tfrac{2}{3}\sqrt{n^3} \quad (n = 1, 2, 3, \ldots). \tag{2}$$

SOLUTION. By Lagrange's theorem,

$$f(x+1)-f(x) = f'(x+\theta) \quad (0 < \theta < 1).$$

From the hypothesis on $f'(x)$ and the preceding equation, it follows that

$$f'(x+1) > f(x+1)-f(x) > f'(x),$$

as was to be proved.

The inequality (1) may be written in the form

$$f(x+1)-f(x) > f'(x) > f(x)-f(x-1). \tag{3}$$

Let $f(x) = \frac{2}{3}x^{3/2}$ and write (3) with x successively replaced by 1, 2, . . ., n. Sum to obtain the inequality (2). (Revised from a solution by Z. Pop-Stojanović.) Generalize the inequality (2).

2.23 Prove the inequality

$$1-x^n > n(1-x)x^{(n-1)/2} \quad (0 < x < 1).$$

2.24 Prove the inequality

$$\frac{n}{x^{-n}-x^n} < \frac{n-1}{x^{-(n-1)}-x^{n-1}} \quad (n \text{ a natural number, } 0 < x < 1).$$

2.25 For what values of x (> 0), p and q is

$$x^p \log^q x \leqq x^{(p-1)/2}?$$

2.26 For what values of x is

$$1 + \frac{x}{2} - \frac{x^2}{8} \leqq (1+x)^{1/2} \leqq 1 + \frac{x}{2}\,? \tag{1}$$

SOLUTION. If $x \geqq -1$, then, by Bernoulli's inequality (§ 0.2),

$$(1+x)^{1/2} \leqq 1 + \frac{x}{2}. \tag{2}$$

For values of x for which

$$1 + \frac{x}{2} - \frac{x^2}{8} < 0 \quad (x \geqq -1) \tag{3}$$

the inequality $1+x/2 - x^2/8 < (1+x)^{1/2}$ holds.

The polynomial $1+x/2-x^2/8$ assumes negative values for x exterior to the interval

$$(-2(\sqrt{3}-1),\ 2(\sqrt{3}+1)). \tag{A}$$

Since, in addition, the condition $x \geqq -1$ is satisfied, we conclude that

$$1+\frac{x}{2}-\frac{x^2}{8} < (1+x)^{1/2} \quad \text{for } x \geqq 2(\sqrt{3}+1). \tag{4}$$

If $x \geqq -1$ and if x is in the interval (A), i.e., if

$$-1 \leqq x \leqq 2(\sqrt{3}+1), \tag{B}$$

then $x+1 \geqq 0$ and $1+x/2-x^2/8 \geqq 0$, and after squaring the inequality $1+x/2-x^2/8 \leqq (1+x)^{1/2}$, we obtain

$$x^3(x-8) \leqq 0$$

which holds for

$$0 \leqq x \leqq 8. \tag{C}$$

Conditions (B) and (C) jointly require that $0 \leqq x \leqq 2(\sqrt{3}+1)$. Collecting these results, we conclude that the inequality

$$1+\frac{x}{2}-\frac{x^2}{8} \leqq (1+x)^{1/2}$$

holds for $x \geqq 0$.

Since, in addition, the inequality (2) holds for $x \geqq -1$, we conclude that (1) holds for $x \geqq 0$.

2.27 Is the inequality $x^2 \geqq 1+2\log x$ true for $x > 0$?

2.28 Show that, if x is real, then

$$\frac{2}{3} \leqq \frac{x^2+1}{x^2+x+1} \leqq 2.$$

Give a geometric interpretation of this result.

2.29 For what values of x is

$$(a-x)^6-3a(a-x)^5+\tfrac{5}{2}a^2(a-x)^4-\tfrac{1}{2}a^4(a-x)^2 < 0?$$

2.30 Prove the inequality

$$\left(\frac{a+1}{b+1}\right)^{b+1} > \left(\frac{a}{b}\right)^b \quad (a,\, b > 0;\ a \neq b). \tag{1}$$

PROOF. Consider the function

$$f(x) = \left(\frac{a+x}{b+x}\right)^{b+x} \quad (x \geqq 0), \tag{2}$$

and its derivative

$$f'(x) = \left(\frac{b-a}{a+x} + \log\frac{a+x}{b+x}\right) f(x). \tag{3}$$

The sign of the derivative is the same as the sign of the function

$$g(x) = \frac{b-a}{a+x} + \log\frac{a+x}{b+x}. \tag{4}$$

Since

$$g'(x) = -\frac{(a-b)^2}{(a+x)^2(b+x)} < 0,$$

the function $g(x)$ is decreasing and, consequently,

$$g(x) > g(+\infty) = 0. \tag{5}$$

On the basis of (5) and (3), we conclude that the function $f(x)$ is increasing, whence

$$\left(\frac{a+x}{b+x}\right)^{b+x} > \left(\frac{a}{b}\right)^{b} \quad (a, b, x > 0;\ a \neq b). \tag{6}$$

The inequality (1) is the special case of (6) obtained when $x = 1$. (Proof by D. Djoković.)

2.31 Prove the inequalities

$$(1-\tfrac{1}{2}x^2)\sin x < (x-\tfrac{1}{6}x^3)\cos x \quad (0 < x < \pi),$$

$$(1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4)\sin x > (x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5)\cos x \quad (0 < x < \pi).$$

What relation holds between the functions

$$\left(\sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}\right) \sin x,\ \left(\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right) \cos x \quad (0 < x < \pi)?$$

2.32 Prove the inequality

$$\cos x < \frac{\sin^2 x}{x^2} \quad (0 < x < \pi/2).$$

SOLUTION. This inequality is equivalent to

$$f(x) > g(x) \quad (0 < x < \pi/2), \tag{1}$$

where $f(x) = \sin^2 x/\cos x$ and $g(x) = x^2$. Without difficulty, we find that

$$f'(x) = \frac{\sin x(1+\cos^2 x)}{\cos^2 x}, g'(x) = 2x,$$

$$f''(x) = \cos x + \frac{1}{\cos x} + \frac{2\sin^2 x}{\cos^3 x}, g''(x) = 2.$$

Since $t + \frac{1}{t} \geqq 2 \ (t > 0)$, $f''(x) > g''(x) = 2 \quad (0 < x < \pi/2)$.

Using this result we find

$$\int_0^x [f''(x)-g''(x)]\, dx = [f'(x)-g'(x)]-[f'(0)-g'(0)]$$
$$= f'(x)-g'(x) > 0,$$

$$\int_0^x [f'(x)-g'(x)]\, dx = [f(x)-g(x)]-[f(0)-g(0)]$$
$$= f(x)-g(x) > 0,$$

which was to be proved.
(Solution by D. Djoković.)

REMARK by D. Adamović. If we consider, instead of the given inequality, the equivalent inequality $x < \sin x/\sqrt{\cos x}$ $(0 < x < \pi/2)$, the proof is simplified.

2.33 1°: Prove the inequality

$$x \log x \geqq x - 1 \quad (x > 0). \tag{1}$$

2°: Starting with this inequality, derive the inequality

$$\sum_{i=1}^{n} p_i \log p_i \geqq \sum_{i=1}^{n} p_i \log q_i, \tag{2}$$

for $p_i > 0$, $q_i > 0$ $(i = 1, 2, \ldots, n)$ and

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i. \tag{3}$$

SOLUTION of 2°. Since $p_i/q_i > 0$, we deduce from (1) that

$$\frac{p_i}{q_i} \log \frac{p_i}{q_i} \geqq \frac{p_i}{q_i} - 1.$$

Since $q_i > 0$, the preceding inequality becomes, after multiplication by q_i,

$$p_i \log \frac{p_i}{q_i} \geqq p_i - q_i.$$

Summing both sides of this inequality over i, one finds

$$\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \geqq \sum_{i=1}^{n} (p_i - q_i).$$

With (3), this inequality gives

$$\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \geqq 0 \Rightarrow \sum_{i=1}^{n} (p_i \log p_i - p_i \log q_i) \geqq 0$$

$$\Rightarrow \sum_{i=1}^{n} p_i \log p_i \geqq \sum_{i=1}^{n} p_i \log q_i,$$

as was to be proved.

Equality holds in (2) if and only if $p_i = q_i$ $(i = 1, 2, \ldots, n)$.

The inequality (2) occurs in Information Theory. See, for example, L. Brillouin: *Science and Information Theory*, New York 1956, pp. 13–14.

2.34 Prove the inequality

$$0 < \sqrt[3]{1+x} - 1 - \tfrac{1}{3}x + \tfrac{1}{9}x^2 < \tfrac{5}{81}x^3 \quad (x > 0).$$

PROOF. Consider

$$f(x) = \sqrt[3]{1+x} - 1 - \tfrac{1}{3}x + \tfrac{1}{9}x^2 \quad (x > 0)$$

and its derivatives

$$f'(x) = \tfrac{1}{3}(1+x)^{-2/3} - \tfrac{1}{3} + \tfrac{2}{9}x,$$

$$f''(x) = \tfrac{2}{9}\left\{1 - \frac{1}{(1+x)^{5/3}}\right\} > 0.$$

Then

$$f'(x) = f'(0)+\int_0^x f''(x)\mathrm{d}x > f'(0) = 0,$$

$$f(x) = f(0)+\int_0^x f'(x)\mathrm{d}x > f(0) = 0.$$

This proves the left-hand side of the inequality.

Next, let

$$g(x) = 1+\tfrac{1}{3}x-\tfrac{1}{9}x^2+\tfrac{5}{81}x^3-\sqrt[3]{1+x} \quad (x > 0),$$

whence

$$g'(x) = \tfrac{1}{3}-\tfrac{2}{9}x+\tfrac{5}{27}x^2-\tfrac{1}{3}(1+x)^{-2/3},$$

$$g''(x) = -\tfrac{2}{9}+\tfrac{10}{27}x+\tfrac{2}{9}(1+x)^{-5/3},$$

$$g'''(x) = \tfrac{10}{27}\left\{1-\frac{1}{(1+x)^{8/3}}\right\} > 0,$$

and so

$$g''(x) = g''(0)+\int_0^x g'''(x)\mathrm{d}x > g''(0) = 0,$$

$$g'(x) = g'(0)+\int_0^x g''(x)\mathrm{d}x > g'(0) = 0,$$

$$g(x) = g(0)+\int_0^x g'(x)\mathrm{d}x > g(0) = 0.$$

This proves the right-hand side of the proposed inequality. (Proof by D. Djoković.)

2.35 If $a > 0$, $b > 0$, $a+b = 1$, then

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geqq \frac{25}{2}.$$

SOLUTION. It follows from $2\sqrt{ab} \leqq a+b = 1$ that

$$1/(ab) \geqq 4. \tag{1}$$

Furthermore

$$L = \left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geqq 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right),$$

and hence

$$-L \leqq -2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right).$$

From this result and (1), we obtain

$$\begin{aligned} L &= \left(a+b+\frac{1}{a}+\frac{1}{b}\right)^2 - 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right) \\ &= \left(1+\frac{1}{ab}\right)^2 - 2\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right) \\ &\geqq (1+4)^2 - L, \end{aligned}$$

i.e.,

$$2L \geqq 25 \Rightarrow L \geqq \tfrac{25}{2}.$$

NOTE. The proof may be obtained in a different manner, using the convex function $f(x) = (x+1/x)^2$ for $x > 0$. In that case we can prove the more general inequality

$$\left(a+\frac{1}{a}\right)^\alpha + \left(b+\frac{1}{b}\right)^\alpha \geqq \frac{5^\alpha}{2^{\alpha-1}} \ (\alpha > 0;\ a, b > 0,\ a+b = 1).$$

The function

$$f(x) = \left(x+\frac{1}{x}\right)^\alpha \quad (\alpha > 0)$$

for $0 < x < 1$ is convex, since

$$\begin{aligned} f''(x) &= \alpha(\alpha-1)\left(x+\frac{1}{x}\right)^{\alpha-2}\left(1-x^{-2}\right)^2 + \frac{2\alpha}{x^3}\left(x+\frac{1}{x}\right)^{\alpha-1} \\ &= \alpha\left(x+\frac{1}{x}\right)^{\alpha-2}\{\alpha(1-x^{-2})^2+x^{-4}-1+4x^{-2}\} > 0 \end{aligned}$$

$(0 < x < 1, \alpha > 0)$.

Consequently, for $a, b > 0$ and $a+b = 1$,

$$\frac{f(a)+f(b)}{2} \geqq f\left(\frac{a+b}{2}\right) = f(\tfrac{1}{2}),$$

i.e.,

$$\left(a+\frac{1}{a}\right)^{\alpha}+\left(b+\frac{1}{b}\right)^{\alpha} \geqq 2(\tfrac{1}{2}+2)^{\alpha}=\frac{5^{\alpha}}{2^{\alpha-1}}.$$

(This solution is due to D. Adamović.)

2.36 Prove that

$$\frac{b-a}{\cos^2 a}<\tan b-\tan a<\frac{b-a}{\cos^2 b} \quad (0 \leqq a<b<\tfrac{1}{2}\pi). \tag{1}$$

PROOF. By the Mean Value Theorem

$$\tan b-\tan a=(b-a)\frac{1}{\cos^2 \theta} \quad (a<\theta<b).$$

Since

$$\frac{1}{\cos^2 a}<\frac{1}{\cos^2 \theta}<\frac{1}{\cos^2 b} \quad (0 \leqq a<\theta<b<\tfrac{1}{2}\pi),$$

(1) follows immediately.

2.37 Prove that

$$1+\frac{1}{n}x-\frac{n-1}{2n^2}x^2<\sqrt[n]{1+x}<1+\frac{1}{n}x \tag{1}$$

$$(n \text{ a natural number} \geqq 2;\ x>0).$$

PROOF. Let us assume that for some x (>0), the inequality

$$\sqrt[n]{1+x}<1+\frac{1}{n}x \quad (x>0) \tag{2}$$

does not hold, but rather that

$$\sqrt[n]{1+x} \geqq 1+\frac{1}{n}x \quad (x>0).$$

Then, raising both members to the n^{th} power, we obtain

$$1+x \geqq \left(1+\frac{1}{n}x\right)^n \equiv 1+\binom{n}{1}\frac{1}{n}x+\binom{n}{2}\frac{1}{n^2}x^2+\ldots+\binom{n}{n}\frac{1}{n^n}x^n. \tag{3}$$

Since

$$\binom{n}{2}\frac{1}{n^2}x^2+\ldots+\binom{n}{n}\frac{1}{n^n}x^n > 0 \quad (x > 0;\ n \text{ an integer} \geqq 2),$$

the relation (3) is false and, consequently, (2) is valid.

Next, consider the function

$$f(x) = \sqrt[n]{1+x}-1-\frac{1}{n}x+\frac{n-1}{2n^2}x^2$$

and its successive derivatives,

$$f'(x) = \frac{1}{n}\left[(1+x)^{(1/n)-1}-1+\frac{n-1}{n}x\right],$$

$$f''(x) = \frac{1-n}{n^2}\left[(1+x)^{(1/n)-2}-1\right],$$

$$f'''(x) = \frac{(1-n)(1-2n)}{n^3}(1+x)^{(1/n)-3}.$$

Since $f'''(x) > 0$, $f''(0) = 0$, $f'(0) = 0$ and $f(0) = 0$ for $x > 0$ and $n \geqq 2$, it follows successively that $f''(x)$, $f'(x)$ and $f(x)$ are increasing functions, whence

$$f(x) > 0 \ (x > 0,\ n \geqq 2).$$

This proves the inequality

$$1+\frac{1}{n}x-\frac{n-1}{2n^2}x^2 < \sqrt[n]{1+x} \quad (x > 0;\ n \text{ integral} \geqq 2). \tag{4}$$

The inequalities (2) and (4) together yield (1).

QUESTION. What happens if $x = 0$ or $n = 1$?

2.38 Prove that

$$x\log x+e^{y-1}-xy \geqq 0 \quad (x > 0).$$

PROOF. Consider the function

$$f(y) = x\log x+e^{y-1}-xy \ \ (x > 0)$$

and its derivatives

$$f'(y) = e^{y-1}-x,\ f''(y) = e^{y-1}.$$

Since $f''(y) > 0$ for all y, the function $f(y)$ attains a minimum of zero for $y = \log x+1$ $(x > 0)$. Consequently

$$f(y) \geqq 0 \quad (x > 0),$$

as was to be proved.

2.39 Prove that

$$1- \frac{x^{n+1}}{(n+1)!} < e^{-x}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right) \quad (x > 0). \qquad (1)$$

PROOF. Consider Taylor's formula,

$$e^x = \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!} + R_n, \text{ where } R_n = \frac{e^{\theta x}}{(n+1)!} x^{n+1} \quad (0 < \theta < 1).$$

Multiplying by e^{-x} one obtains

$$1 = e^{-x}\left(\sum_{\nu=0}^{n} \frac{x^\nu}{\nu!}\right) + \frac{e^{(\theta-1)x}}{(n+1)!} x^{n+1}. \qquad (2)$$

Since $0 < \theta < 1$ and $x > 0$, $1 > e^{(\theta-1)x}$, and thus

$$\frac{x^{n+1}}{(n+1)!} > \frac{e^{(\theta-1)x}}{(n+1)!} x^{n+1}. \qquad (3)$$

Now (1) follows from (2) and (3).

2.40 Prove that a function $f(x)$ having a second derivative has no zero in the interval (a, b) if there exists $c \in (a,b)$ such that for every $x \in (a, b)$

$$f'^2(c)-2f(c)f''(x) < 0. \qquad (1)$$

SOLUTION 1. Let us assume that the converse statement is true. If $y \in (a, b)$ is a zero of $f(x)$ and $c \in (a, b)$ satisfies the given condition of our problem, then we have the following:

$$f(y) = f(c)+\frac{(y-c)}{1!} f'(c)+\frac{(y-c)^2}{2!} f''(x) = 0, \; x \in (a, b),$$

$$f(c) + \alpha f'(c) + \alpha^2 \frac{f''(x)}{2} = 0 (\alpha = y - c), \text{ which implies:}$$

$$f'^2(c) - 2f(c)f''(x) \geqq 0.$$

From this contradiction it follows that $f(x)$ has no zero in the interval (a, b).

SOLUTION 2. From (1) it follows that $f(c) \neq 0$. For $x \neq c$ we obtain

$$2f(c)f'(x) - 2f(c)f'(c) = 2f(c)f''(\xi)(x-c) \gtrless (x-c)f'^2(c)$$

according as $x \gtrless c$. In both cases

$$2f(c)f(x) = 2f^2(c) + 2f(c)\int_c^x f'(x)\mathrm{d}x$$

$$> 2f^2(c) + \int_c^x [2f(c)f'(c) + (x-c)f'^2(c)]\mathrm{d}x$$

$$= \tfrac{1}{2}[2f(c) + (x-c)f'(c)]^2.$$

Hence, $f(x)$ cannot vanish.
(Problem and solution 1 by S. Prešić. Solution 2 by D. Djoković.)

2.41 $|a \sin x + b \cos x| \leqq \sqrt{a^2+b^2}$.

2.42 $\cos^4 x + \sin^4 x \geqq \tfrac{1}{2}$.

2.43 $\dfrac{e^x + e^y}{2} > e^{\frac{1}{2}(x+y)} \quad (x \neq y)$.

2.44 $2\sqrt{x} > 3 - \dfrac{1}{x} \quad (x > 1)$.

2.45 $0 \leqq (x+a)^2/(x^2+x+1) \leqq \tfrac{4}{3}(a^2-a+1)$.

2.46 $x_2^{1/n} - x_1^{1/n} \leqq (x_2-a)^{1/n} - (x_1-a)^{1/n} \quad (0 \leqq a \leqq x_1 \leqq x_2)$.

2.47 $\dfrac{1}{2^{p-1}} \leqq x^p + (1-x)^p \leqq 1 \quad (0 \leqq x \leqq 1;\ p > 1)$.

2.48 $y(a^{1/y}-1) < x(a^{1/x}-1) \quad (a > 0;\ a \neq 1;\ 0 < x < y)$.

2.49 $\dfrac{1-(\sin x)^{p-1}}{p-1} > \dfrac{1-(\sin x)^p}{p} \quad \left(p > 1;\ 0 < x < \dfrac{\pi}{2}\right)$.

2.50 $\dfrac{x^m-1}{m} < \dfrac{x^n-1}{n} \quad (m < n;\ x > 0 \text{ and } x \neq 1)$.

2.51 $1 \geqq \dfrac{1+2x+\ldots+nx^{n-1}}{1+2^2x+\ldots+n^2x^{n-1}} \geqq \dfrac{1}{n} \quad (x > 0).$

2.52 $n \geqq \dfrac{1+2x+\ldots+nx^{n-1}}{n+(n-1)x+\ldots+x^{n-1}} \geqq \dfrac{1}{n} \quad (x > 0).$

2.53 $(k+1)\cos\dfrac{\pi}{k+1} > 1+k\cos\dfrac{\pi}{k} \quad (k \geqq 2).$

2.54 $(e+x)^{e-x} > (e-x)^{e+x} \quad (0 < x < e).$

2.55 $x-\frac{1}{2}x^2 < \log(1+x) < x \quad (x > 0).$

2.56 $2x/(2+x) < \log(1+x) \quad (x > 0).$

2.57 $x^2 > (1+x)\log^2(1+x) \quad (x > -1).$

2.58 $|\log(1+x)-x| \leqq x^2 \quad (|x| \leqq \frac{1}{2}).$

2.59 $1+x\log(x+\sqrt{1+x^2}) \geqq \sqrt{1+x^2}.$

2.60 $\log(1+x) > (\arctan x)/(1+x) \quad (x > 0).$

2.61 $2x\arctan x \geqq \log(1+x^2).$

2.62 $(x+1)\log(x+1)-x\log x > 0 \quad (x > 0).$

2.63 $x^3+3x+2+6x\log x > 6x^2 \quad (x > 1).$

2.64 $x^a|\log x| \leqq 1/(ae) \quad (0 < x < 1; \quad a > 0).$

2.65 $|\log(1+x)| \leqq \dfrac{|x|(1+|x|)}{|x+1|} \quad (x > -1).$

2.66 $\log\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}+\dfrac{1}{x+1}-\dfrac{1}{x+\frac{1}{2}} \quad (x > 0).$

2.67 $\dfrac{\log x}{x-1} \leqq \dfrac{1+x^{1/3}}{x+x^{1/3}} \quad (x > 0 \text{ and } x \neq 1).$

2.68 $x < \dfrac{1}{1-\dfrac{x}{2}} < -\log(1-x) < \dfrac{x}{1-x} \quad (0 < x < 1).$

2.69 $\dfrac{1}{x+\dfrac{1}{2}} < \log\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x} \quad (x > 0).$

2.70 $\dfrac{1}{x+1}+\dfrac{1}{2(x+1)^2} < \log\left(1+\dfrac{1}{x}\right) < \dfrac{1}{2x^2}+\dfrac{1}{x+1} \quad (x > 0)$

2.71 $\log\dfrac{b}{a} < \log\dfrac{1+b}{1+a} \quad (a > b > 0).$

2.72 $e^x > \dfrac{1}{k!}x^k \quad (x > 0;\, k = 0, 1, 2, \ldots).$

2.73 $\dfrac{x}{1-x} > 1-e^{-x} \quad (x < 1;\, x \neq 0).$

2.74 $\dfrac{1+x}{1-x} > e^{2x} \quad (0 < x < 1).$

2.75 $\exp\left(-\dfrac{x}{1-x}\right) < 1-x \quad (x < 1;\, x \neq 0).$

2.76 $ae^{-b\theta}-be^{-a\theta} < a-b \quad (a > b > 0;\, \theta > 0).$

2.77 $\exp x > \left(1+\dfrac{x}{y}\right)^y > \exp\dfrac{xy}{x+y} \quad (x, y > 0).$

2.78 $\log(1+\sqrt{1+x^2}) < \dfrac{1}{x}+\log x \quad (x > 0).$

2.79 $\dfrac{a-b}{a} < \log a-\log b < \dfrac{a-b}{b} \quad (0 < b < a).$

2.80 $e^x < (1+x)^{1+x} \quad (x > 0).$

2.81 $e^{2x} > 1/(1-x) \quad (0 < x < 1/2).$

2.82 $xe^{x/2} < e^x - 1 < xe^x \quad (x > 0).$

2.83 $e^{x^2/2}\cos x < 1 \quad (0 < x \leqq 4).$

2.84 $\cosh x \cos x < 1 \quad (0 < x \leqq \pi/2).$

2.85 $0 \leqq e^{-x}\left(1-\dfrac{x}{n}\right)^n \leqq \dfrac{1}{2n}x^2 \quad (x > 0).$

2.86 $|e^x(12-6x+x^2)-(12+6x+x^2)| \leqq \frac{1}{60}|x|e^{|x|}$.

2.87 $x^4+8x+12x^2 \log x > 8x^3+1 \quad (x > 1)$.

2.88 $1-\sqrt[n]{1-x} \leqq \frac{1}{n}|\log(1-x)| \leqq \frac{x}{2n}\frac{2-x}{1-x} \quad (0 < x < 1)$.

2.89 $\log(1+\cos x) < \log 2-\frac{1}{4}x^2 \quad (0 < x < \pi)$.

2.90 $\log(1+x^2) < x \arctan x < x^2 \quad (x > 0)$.

2.91 $ry^{r-1} < \frac{x^r-y^r}{x-y} < rx^{r-1} \quad (x > y > 0;\ r > 1)$.

2.92 $rx^{r-1} < \frac{x^r-y^r}{x-y} < ry^{r-1} \quad (x > y > 0;\ 0 < r < 1)$.

§ 3. Inequalities Involving Powers and Factorials

3.1 Prove the inequalities:

1°: $n^n > (2n-1)!!$, 2°: $(n+1)^n > (2n)!!$ $(n > 1)$.

Solution 1. 1°: For $n = 2$, the inequality

$$n^n > (2n-1)!! \tag{1}$$

is valid.

Assuming that (1) holds for $n = k$, i.e.,

$$k^k > (2k-1)!!, \tag{2}$$

then the inequality obtained by multiplying (2) by $(k+1)^{k+1}/k^k$ is also valid; thus,

$$(k+1)^{k+1} > (2k-1)!!\ (k+1)^{k+1}/k^k. \tag{3}$$

If

$$(2k-1)!!\ (k+1)^{k+1}/k^k > (2k+1)!!, \tag{4}$$

then

$$(k+1)^{k+1} > (2k+1)!! \tag{5}$$

The inequality (4) is valid. Indeed, instead of (4), we may consider

$$(k+1)^{k+1}-(2k+1)k^k > 0. \tag{6}$$

After application of the binomial expansion, the left-hand side of (6) becomes

$$\binom{k+1}{2} k^{k-1} + \binom{k+1}{3} k^{k-2} + \ldots + \binom{k+1}{k} k + \binom{k+1}{k+1},$$

which is positive for all natural numbers k.

From hypothesis (2), i.e., $f(k) > g(k)$, we have shown that $f(k+1) > g(k+1)$. Since, in addition, (1) holds for $n = 2$, it follows that it holds for every $n \in \{2, 3, 4, \ldots\}$.

The second inequality may be proved in a similar manner.

SOLUTION 2. 1°: From $(a-b)^2 \geqq 0$, it follows that

$$a^2 \geqq b(2a-b), \tag{1}$$

with inequality if $a \neq b$.

If we let $a = n$ and $b = 2k-1$ $(k = 1, 2, \ldots, n)$, we have successively

$$n^2 \geqq 1 \cdot (2n-1),$$
$$n^2 \geqq 3 \cdot (2n-3),$$
$$\ldots$$
$$n^2 > (2n-1) \cdot 1.$$

Multiplication of these inequalities yields

$$n^{2n} > \{(2n-1)!!\}^2 \Rightarrow n^n > (2n-1)!!$$

2°: If in (1) we set $a = n+1$, $b = 2k$ $(k = 1, 2, \ldots, n)$, we obtain

$$(n+1)^2 \geqq 2 \cdot 2n,$$
$$(n+1)^2 \geqq 4\ (2n-2),$$
$$\ldots$$
$$(n+1)^2 > 2n \cdot 2.$$

After multiplication, we obtain

$$(n+1)^{2n} > \{(2n)!!\}^2 \Rightarrow (n+1)^n > (2n)!!$$

(Solution by B. Mesihović.)

3.2 Prove the inequality

$$(n+1)^{n-1}(n+2)^n > 3^n(n!)^2 \quad (n > 1).$$

HINT. Use the inequality $(n+1)^n > (2n)!!$ $(n > 1)$ of the preceding exercise and the inequality $2^{2n+1} > 3^n$.

3.3 Prove the inequality

$$n! < \left(\frac{n+1}{2}\right)^n \quad (n \text{ a natural number} > 1). \tag{1}$$

PROOF. We start with the inequality

$$\left(\frac{k+2}{k+1}\right)^{k+1} = \left(1+\frac{1}{k+1}\right)^{k+1} > 2 \ (k = 1, 2, 3, \ldots).$$

Setting $k = 1, 2, 3, \ldots, n-1$, we obtain the inequalities:

$$\left(\frac{3}{2}\right)^2 > 2, \left(\frac{4}{3}\right)^3 > 2, \ldots, \left(\frac{n}{n-1}\right)^{n-1} > 2, \left(\frac{n+1}{n}\right)^n > 2.$$

After multiplication, we obtain

$$\frac{(n+1)^n}{2^2 \cdot 3 \cdot 4 \ldots n} > 2^{n-1},$$

whence the inequality (1) follows immediately.

3.4. Prove the inequality

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2} \quad (n = 2, 3, 4, \ldots). \tag{1}$$

SOLUTION. For $n = 2$, the relation (1) certainly holds, because

$$16/3 < 4!/(2!)^2 = 6.$$

Suppose that (1) is valid for some natural number $n = k$ $(\geqq 2)$, that is to say

$$4^k/(k+1) < (2k)!/(k!)^2,$$

or, alternatively,

$$(2k)!/(k!)^2 > 4^k/(k+1). \tag{2}$$

After multiplying both sides of (2) by

$$(2k+1)(2k+2)/(k+1)^2,$$

we have

$$(2k+2)!/\{(k+1)!\}^2 > (2k+1)(2k+2)4^k/(k+1)^3,$$

which may be written in the form

$$\frac{(2k+2)!}{\{(k+1)!\}^2} > \frac{4^{k+1}}{k+2}\,\frac{(k+2)(2k+1)}{2(k+1)^2}.$$

Since

$$\frac{(k+2)(2k+1)}{2(k+1)^2} = 1 + \frac{k}{2(k+1)^2} > 1 \ (k > 0),$$

we have

$$\frac{(2k+2)!}{\{(k+1)!\}^2} > \frac{4^{k+1}}{k+2}.$$

The proof of the inequality (1) now follows by induction.

3.5 Prove the inequality

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{n}}.$$

SOLUTION. Let

$$N = \frac{(2n-1)!!}{(2n)!!} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}.$$

From the inequalities

$$\frac{2k-1}{2k} < \frac{2k}{2k+1} \quad (k = 1, 2, 3, \ldots)$$

it follows that

$$N < \frac{2\cdot 4\cdot 6\ldots(2n)}{3\cdot 5\cdot 7\ldots(2n+1)} = 1\Big/\frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n)}$$
$$= \frac{1}{(2n+1)N} < \frac{1}{nN}.$$

Hence

$$N^2 < 1/n \Rightarrow N < 1/\sqrt{n}.$$

3.6 Prove the inequality

$$2!\,4!\ldots(2n)! > \{(n+1)!\}^n \quad (n \text{ a natural number} \geqq 2). \quad (1)$$

SOLUTION. For $n = 2$, the inequality (1) is valid. Let us suppose now that it holds for $n = k-1$, i.e.,

$$2!\,4!\ldots(2k-2)! > (k!)^{k-1}.$$

Then,

$$2!\,4!\ldots(2k-2)!\,(2k)! > (2k)!(k!)^{k-1}$$
$$= \frac{(2k)!(k!)^k}{k!} = (2k)(2k-1)\ldots(k+1)(k!)^k$$
$$> (k+1)^k(k!)^k = \{(k+1)!\}^k,$$

because each of the factors $2k$, $2k-1, \ldots, k+2$ is greater than $k+1$. Consequently, the inequality (1) is proved by induction.

3.7 Prove that $n! > n^{n/2}$ (n a natural number > 2).

PROOF. $n! = 1\cdot 2\ldots n$, $(n!)^2 = 1^2\cdot 2^2\ldots n^2$, which may be written in the form

$$(n!)^2 = \{1\cdot n\}\{2(n-1)\}\{3(n-2)\}\ldots\{r(n-r+1)\}\ldots\{n\cdot 1\}, \quad (1)$$

where $r(1 \leqq r \leqq n)$ is a natural number. For all such r,

$$r(n-r+1) \geqq n, \text{ since } (r-1)(r-n) \leqq 0. \quad (2)$$

If we set $r = 1, 2, \ldots, n$ successively, we obtain from (2)

$$1\cdot n = n;\ 2(n-1) > n;\ \ldots;\ r(n-r+1) > n;\ \ldots;\ n\cdot 1 = n.$$

Hence it follows from (1) that

$$(n!)^2 > n^n \ (n = 3, 4, \ldots),$$
$$n! > n^{n/2} \ (n = 3, 4, \ldots).$$

3.8 Prove the inequality

$$2^{n(n-1)/2} > n! \quad (n \text{ a natural number } > 2).$$

HINT. Write $2^{n(n-1)/2}$ in the form

$$2^{1+2+\ldots+(n-1)} = 2^1 \cdot 2^2 \ldots 2^{n-1},$$

and use the inequality $2^n > n+1$ $(n \geqq 2)$ which may be proved by the method of complete induction.

3.9 Show that if $x_1, x_2, \ldots, x_n$ are natural numbers satisfying the equation

$$x_1+x_2+\ldots+x_n = np \quad (p \text{ a natural number}),$$

then

$$x_1!+x_2!+\ldots+x_n! \geqq n(p!).$$

3.10 If both sides of the inequality

$$2 \sin x - \sin^2 x \leqq 1 \quad (\Leftrightarrow (1-\sin x)^2 \geqq 0)$$

are multiplied by

$$\sin^{2m-1} x \quad (0 < x < \pi/2, \ m \text{ a natural number})$$

and if the inequality thus obtained is integrated between the limits $(0, \pi/2)$, one obtains the inequality

$$2\int_0^{\pi/2} \sin^{2m} x \, dx - \int_0^{\pi/2} \sin^{2m+1} x \, dx < \int_0^{\pi/2} \sin^{2m-1} x \, dx.$$

Show that from this result we may derive

$$\left\{\frac{2\pi m(2m+1)}{4m+1}\right\}^{1/2} < \frac{(2m)!!}{(2m-1)!!}.$$

3.11 Starting with the inequalities

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x \quad (0 < x < \pi/2),$$

integrate between the limits $(0, \pi/2)$ to prove Wallis' inequality

$$\frac{2}{2n+1}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2 < \pi < \frac{1}{n}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2.$$

3.12 Prove the inequality

$$\frac{4n+3}{(2n+1)^2}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2 < \pi < \frac{4}{4n+1}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2.$$

NOTE. This stronger Wallis' formula was given by J. Gurland in *The American Mathematical Monthly*, vol. 63, 1956, pp. 643-645.

3.13 Prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \leqq \frac{1}{(3n+1)^{1/2}} \quad (n \text{ a natural number}). \qquad (1)$$

SOLUTION. For $n = 1$, the relation is valid. Let us suppose that it holds for $n = k$; i.e., that

$$f(k) \equiv \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k} \leqq \frac{1}{(3k+1)^{1/2}} \equiv g(k).$$

Multiply both sides of the preceding inequality by the positive number $(2k+1)/(2k+2)$. Then we have

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} \leqq \frac{1}{(3k+1)^{1/2}}\frac{2k+1}{2k+2}.$$

If we can show that

$$\frac{1}{(3k+1)^{1/2}}\frac{2k+1}{2k+2} \leqq \frac{1}{[3(k+1)+1]^{1/2}}, \qquad (2)$$

then the hypothesis $f(k) \leqq g(k)$ implies that $f(k+1) \leqq g(k+1)$.

Let us suppose that the relation (2) is not valid, i.e., that

$$(2k+1)/\{(3k+1)^{1/2}(2k+2)\} > 1/[3(k+1)+1]^{1/2}.$$

Then we find that

$$\frac{1}{3k+1}\frac{(2k+1)^2}{(2k+2)^2} > \frac{1}{3k+4},$$

$$12k^3+28k^2+19k+4 > 12k^3+28k^2+20k+4,\ 19k > 20k,$$

which cannot be valid when $k > 0$. Consequently, the relation (2) is valid. By induction, this proves (1).

3.14 If $n = \sum_{\nu=1}^{k} n_\nu$ (n_ν natural numbers), then

$$n!\Big/\prod_{\nu=1}^{k}(n_\nu!) \leqq n^n\Big/\prod_{\nu=1}^{k}(n_\nu^{n_\nu}). \tag{1}$$

PROOF. Since every term in the expansion of $(n_1+n_2+\ldots+n_k)^n$ is positive and less than the sum of all the terms,

$$\frac{n!}{\prod_{\nu=1}^{k}(n_\nu!)}\prod_{\nu=1}^{k}(n_\nu^{n_\nu}) \leqq n^n \Leftrightarrow (1).$$

Equality holds in (1) if $n = n_1$.

This solution is by Chih-yi Wang (*Mathematics Magazine*, vol. 31, No. 2, 1957, p. 113).

3.15 If $x_r = x(x-1)(x-2)\ldots(x-r+1)$ (r a natural number), $x_0 = 1$, prove the inequality

$$(2n-1)_{n-1}(2n-3)_{n-3}(2n-5)_{n-5}\ldots > n_{n-1}n_{n-3}n_{n-5}\ldots,$$

where the products on both sides extend over all non-negative indices of the same parity as $n-1$, and $n > 1$.

SOLUTION. Since $x_r = \binom{x}{r}r!$, if x is a positive integer, the products on the left and right-hand sides may be written in the forms

$$L = \binom{2n-1}{n-1}(n-1)!\binom{2n-3}{n-3}(n-3)!\binom{2n-5}{n-5}(n-5)!\ldots$$

$$= \prod_{\nu=0}^{[\frac{1}{2}(n-1)]}\binom{2n-2\nu-1}{n-2\nu-1}(n-2\nu-1)!,$$

$$R = \binom{n}{n-1}(n-1)!\binom{n}{n-3}(n-3)!\binom{n}{n-5}(n-5)!\ldots$$
$$= \prod_{\nu=0}^{[\frac{1}{2}(n-1)]} \binom{n}{n-2\nu-1}(n-2\nu-1)!.$$

From these expressions we have

$$\frac{R}{L} = \prod_{\nu=0}^{[\frac{1}{2}(n-1)]} \frac{\binom{n}{n-2\nu-1}}{\binom{2n-2\nu-1}{n-2\nu-1}} = \prod_{\nu=0}^{[\frac{1}{2}(n-1)]} \frac{n!\,n!}{(2n-2\nu-1)!\,(2\nu+1)!}$$
$$= \prod_{\nu=0}^{[\frac{1}{2}(n-1)]} \frac{n!\,n!}{(2n)!}\binom{2n}{2\nu+1}.$$

From the characteristic symmetry of the binomial coefficients and the fact that the middle coefficient of an expansion is the greatest, we have

$$\binom{2n}{2\nu+1} < \binom{2n}{n} \quad \text{for } \nu \neq \tfrac{1}{2}(n-1).$$

Because $\binom{2n}{n} = (2n)!/(n!)^2$, we thus have $R/L < 1$, i.e., $L > R$, which proves the given inequality.
(This solution is by S. Prešić).

3.16 Use mathematical induction to prove the inequality

$$n \log n - n < \log n! < (n+\tfrac{1}{2})\log n - n + 1 \tag{1}$$

(n a natural number > 1).

Solution. We set

$$u(n) = n \log n - n,\ v(n) = (n+\tfrac{1}{2})\log n - n + 1.$$

Let us first prove the inequality

$$u(n) < \log n! \tag{2}$$

This is valid for $n = 1$. Let us assume that (2) holds for some natural number $n = k$, i.e., that

$$u(k) < \log k!; \tag{3}$$

then

$$u(k)+\log(k+1) < \log k!+\log(k+1) = \log(k+1)!.$$

If we prove that

$$(k+1)\log(k+1)-(k+1) < k\log k-k+\log(k+1), \tag{4}$$

then we shall have established that

$$u(k+1) < \log\,(k+1)!,$$

and thus that the relation (2) is valid for $n = k+1$, if it is valid for $n = k$.

Let us suppose that (4) is false for some k, so that

$$(k+1)\log(k+1)-(k+1) \geqq k\log k-k+\log(k+1). \tag{5}$$

Then

$$k\log\frac{k+1}{k} \geqq 1 \Rightarrow \log\left(1+\frac{1}{k}\right)^k \geqq 1,$$

whence it follows that

$$\left(1+\frac{1}{k}\right)^k \geqq e. \tag{6}$$

Thus, starting with (5), we arrive at (6), which is false for all k. This proves the inequality (4), and thereby (2).

Next, consider the inequality

$$\log n! < (n+\tfrac{1}{2})\log n-n+1 = v(n). \tag{7}$$

For $n = 2$, we have $\log 2! < \frac{5}{2}\log 2-1$, since $\log 2 = 0.6931\ldots$. Let us now suppose that (7) is valid for $n = k$, i.e., that

$$\log k! < v(k) \quad (k > 1). \tag{8}$$

It then follows that

$$\log k!+\log(k+1) < (k+\tfrac{1}{2})\log k-k+1+\log(k+1).$$

If we show that

$$(k+\tfrac{1}{2})\log k-k+1+\log(k+1) < (k+\tfrac{3}{2})\log(k+1)-(k+1)+1, \tag{9}$$

then we shall have shown that the inequality (7) holds for $n = k+1$, if it holds for $n = k$.

The inequality (9) may be written in the form

$$(k+\tfrac{1}{2}) \log \frac{k}{k+1} + 1 < 0,$$

or

$$\log \frac{k}{k+1} + \frac{2}{2k+1} < 0. \tag{10}$$

Setting

$$f(x) = \log \frac{x}{x+1} + \frac{2}{2x+1} \quad (x > 0),$$

we find that

$$f'(x) = \frac{1}{x(x+1)(2x+1)^2}.$$

For $x > 0$, it follows that $f'(x)$ is positive, whence $f(x)$ is an increasing function for positive x. As $x \to +\infty$, $f(x) \to 0$. Hence $f(x) < 0$ for $x > 0$.

This completes the proof of (10), and the proof of (1) follows by induction.

3.17 Prove the inequality

$$n^{n+1} > (n+1)^n \quad (n \text{ a natural number} > 2). \tag{1}$$

Proof. For $n = 3$, (1) is true. Let us suppose that (1) holds for $n = k$, i.e., that

$$k^{k+1} > (k+1)^k \quad (k > 2). \tag{2}$$

Since

$$(k+1)^2 > k(k+2) \Rightarrow \frac{k+1}{k} > \frac{k+2}{k+1},$$

it follows that

$$\left(\frac{k+1}{k}\right)^{k+1} > \left(\frac{k+2}{k+1}\right)^{k+1}. \tag{3}$$

From (2) and (3), after multiplying, we obtain

$$(k+1)^{k+2} > (k+2)^{k+1},$$

whence the truth of (1) follows by induction.

3.18 Prove J. van Hengel's inequality

$$(n+r)^n < n^{n+r} \quad (n, r \text{ natural numbers}, n \geqq 3 \text{ or } r \geqq 3).$$

REMARK. Concerning this inequality see: Dickson: *History of the Theory of Numbers*, vol. 2, p. 687.

3.19 Prove Goormaghtigh's inequality

$$(n+r)^n < n^{r-p}(n+p)^n \quad (n, p, r \text{ natural numbers}, n \geqq 3, p < r).$$

REMARK. See: *Mathesis*, vol. 68, 1959, p. 374–375.

3.20 $\log n! > (n+\frac{1}{2}) \log n - n + \dfrac{1}{12n}$.

3.21 $\frac{1}{2}(n+1) < (1^1 2^2 3^3 \ldots n^n)^{2/(n^2+n)} < \frac{1}{3}(2n+1) \quad (n > 1)$.

3.22 $2^n > 1+n\sqrt{2^{n-1}} \quad (n > 1)$.

3.23 $\sqrt[n]{n+1} < \sqrt[n-1]{n} \quad (n \geqq 2)$.

3.24 $n! \leqq \left(\dfrac{n}{2}\right)^n \quad (n \geqq 6)$.

3.25 $(n!)^3 \leqq n^n \left(\dfrac{n+1}{2}\right)^{2n}$.

3.26 $n < \{(n+1)^{1+1/n}(n-1)^{1-1/n}\}^{1/2} < n+1/n \quad (n > 1)$.

3.27 $\dfrac{1}{2\sqrt{k}} < \dfrac{1}{4^k}\dbinom{2k}{k} < \dfrac{1}{\sqrt{3k+1}} \quad (k > 1)$.

3.28 $\dfrac{3 \cdot 7 \cdot 11 \ldots (4n-1)}{5 \cdot 9 \cdot 13 \ldots (4n+1)} < \sqrt{\dfrac{3}{4n+3}}$.

3.29 $\dfrac{1}{2\sqrt{n}} < \dfrac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} < \dfrac{1}{\sqrt{2n+1}} \quad (n > 1)$.

3.30 $\frac{3n-1}{2n} < \left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n-1}{n^2}\right) < \frac{2n}{n+1}$ $(n > 2)$.

3.31 $\frac{4\cdot 7\cdot 10\ldots(3n+4)}{3\cdot 5\cdot 8\ldots(3n+2)} > 1+\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right)$.

3.32 $a+b+c \leqq ad^{b-c}+bd^{c-a}+cd^{a-b}$ $(a, b, c, d > 0)$.

3.33 $a^c b^d (c+d)^{c+d} \leqq c^c d^d (a+b)^{c+d}$ $(a, b, c, d > 0)$.

3.34 $a^a b^b (a+b)^{a+b} \leqq (a^2+b^2)^{a+b}$ $(a, b > 0)$.

3.35 $a^{2a} < (a+b)^{a+b}(a-b)^{a-b} < \left(a+\frac{b^2}{a}\right)^{2a}$ $(0 < b < a)$.

3.36 $a^b+b^a > 1$ $(a, b > 0)$.

3.37 $a^a b^b > a^b b^a$ $(a > b > 0)$.

3.38 $p^{p/(p+q+r)}q^{q/(p+q+r)}r^{r/(p+q+r)} \geqq \frac{1}{3}(p+q+r)$ (p, q, r natural numbers).

3.39 $(1+a)^{1-a}(1-a)^{1+a} < 1 < (1+a)^{1+a}(1-a)^{1-a}$ $(0 < a < 1)$

3.40 $|a^{m-1}(a-1)^m \ldots (a-n)^m| \leqq (n!)^m$ $(0 \leqq a \leqq n)$.

3.41 $\sqrt[n]{n} \leqq 1+\sqrt{\frac{2}{n}}$.

3.42 $\frac{2^n}{n} < \frac{n^n}{n!} < 3^{n-1}$ $(n \geqq 3)$.

3.43 $n^n \geqq (2n-1)!! \geqq (2n-1)^{n/2}$.

3.44 $\binom{m}{n} \geqq \left(\frac{2m-n+1}{n+1}\right)^n$.

3.45 $(a^p-b^p)(a^q+b^q) > (a^q-b^q)(a^p+b^p)$ $(a > b > 0,\ p > q)$

3.46 $(pa+qb)^{p+q} > (p+q)^{p+q}\cdot a^p b^q$ $(a, b, p, q > 0, p \neq q)$.

3.47 $a^c b^{1-c}+(1-a)^c(1-b)^{1-c} \leqq 1$ $(0 < a, b, c < 1)$.

3.48 $\left(\frac{x+1}{2}\right)^{x+1} \leqq x^x \quad (x > 0).$

3.49 $a^a b^b > \left(\frac{a+b}{2}\right)^{a+b} \quad (a, b > 0, a \neq b).$

3.50 $\left(\frac{a^4+b^4+c^4}{a+b+c}\right)^{(a+b+c)/2} \geqq a^a b^b c^c \quad (a, b, c > 0).$

3.51 $(bcd+cda+dab+abc)^{a+b+c+d}$
$\geqq a^{b+c+d-a} b^{c+d+a-b} c^{d+a+b-c} d^{a+b+c-d} \, (a+b+c+d)^{a+b+c+d}$
$(a, b, c, d > 0)$

3.52 $\left(1+\frac{1}{m}\right)^m > \left(1+\frac{1}{n}\right)^n > 2^n \left(\frac{n!}{n^n}\right)^{3/2} \quad (m > n > 1).$

3.53 $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{2n}\right)^{2n+1}.$

3.54 $\left(1+\frac{1}{n^2}\right)^{2n} > \left(1-\frac{1}{n}\right)^{n-1} \left(1+\frac{1}{n}\right)^{n+1}.$

3.55 $2ab(c+1) \leqq a^{c+1}+b^{c+1}+c(a^{1+1/c}+b^{1+1/c}) \quad (a, b, c > 0).$

3.56 $n^n \left(\frac{n+1}{2}\right)^{3n} > (n!)^4 \quad (n > 1).$

3.57 $m!+n! > (m-1)!+(n+1)! \quad (m-n > 1).$

3.58 $\prod_{k=0}^{n} \binom{n}{k} \leqq \left(\frac{2^n-2}{n-1}\right)^{n-1} \quad (n \geqq 2).$

3.59 $\frac{a^p}{b^{p-1}} \geqq pa-(p-1)b \quad (a, b > 0, p > 1).$

3.60 $\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n < \left(1+\frac{1}{n}\right)e.$

3.61 $\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^{a_1+a_2+\cdots+a_n} \leqq a_1^{a_1} a_2^{a_2} \ldots a_n^{a_n} \quad (a_i > 0)$

3.62 $2 < \left(1+\frac{1}{m}\right)^m < \left(1+\frac{1}{n}\right)^n < 3 \quad (1 < m < n).$

3.63 $\left(1+\frac{a}{m}\right)^m < \left(1+\frac{a}{n}\right)^n \quad (a > 0;\, m < n).$

3.64 $\left(1-\frac{a}{x}\right)^{-x} > \left(1-\frac{a}{y}\right)^{-y} \quad (0 < a < x < y).$

3.65 $(1+a^t)^{1/t}-(1+a^t)^{-1/t} \leqq a \quad (a \geqq 0,\, t \geqq 2).$

§ 4. Inequalities Involving Finite Sums and Products

4.1 Prove the inequality

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1) \quad (n \geqq 1). \quad (1)$$

SOLUTION 1. For $n = 1$, the inequality (1) certainly holds. Let us suppose that (1) is valid for $n = k$, i.e., that

$$1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} > 2(\sqrt{k+1}-1).$$

Then

$$1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1}-1)+\frac{1}{\sqrt{k+1}}.$$

If we can show that

$$2(\sqrt{k+1}-1)+(1/\sqrt{k+1}) > 2(\sqrt{k+2}-1) \quad (k \geqq 1), \quad (2)$$

then the proof of (1) will follow by induction.

Now (2) is equivalent to

$$2\sqrt{k+1}+(1/\sqrt{k+1}) > 2\sqrt{k+2}.$$

If, on the contrary,

$$2\sqrt{k+1}+(1/\sqrt{k+1}) \leqq 2\sqrt{k+2}, \quad (3)$$

then

$$2k+3 \leqq 2\sqrt{(k+1)(k+2)} \Rightarrow 9 \leqq 8.$$

Since hypothesis (3) leads to a contradiction, it follows that the inequality (2) holds for $k \geqq 1$.

SOLUTION 2. Since the function $f(x) = 1/\sqrt{x}$ is monotone decreasing,

$$\sum_{\nu=1}^{n} f(\nu) > \int_1^{n+1} f(x)\mathrm{d}x \text{ or } \sum_{\nu=1}^{n} \frac{1}{\sqrt{\nu}} > 2(\sqrt{n+1}-1),$$

which was to be proved.

4.2 Prove the inequality

$$\sum_{k=0}^{m} \frac{1}{(m+k)(m+k-1)} > \sum_{k=m}^{2m} \frac{1}{k^2} > \sum_{k=0}^{m} \frac{1}{(m+k)(m+k+1)} \quad (m > 1).$$

4.3 Prove the inequality

$$f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \tfrac{1}{2} \quad (n > 1).$$

SOLUTION. For $1 \leqq \nu \leqq n$, $n+\nu \leqq 2n$. Thus, $1/(n+\nu) \geqq 1/2n$ for each of the n terms of $f(n)$, whence $f(n) > n/(2n) = 1/2$.

4.4 Prove the inequality

$$f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n+1} > 1 \quad (n = 1, 2, 3, \dots).$$

SOLUTION. Let us suppose that the given inequality is valid for some natural number $n = k$, that is to say

$$f(k) > 1. \tag{1}$$

If both sides of this inequality are increased by

$$-\frac{1}{k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4},$$

we obtain

$$f(k+1) > 1 - \frac{2}{3k+3} + \frac{1}{3k+2} + \frac{1}{3k+4}. \tag{2}$$

If

$$1 - \frac{2}{3k+3} + \frac{1}{3k+2} + \frac{1}{3k+4} > 1, \tag{3}$$

it will follow from the induction hypothesis that $f(k+1) > 1$.

Now (3) is equivalent to

$$\frac{1}{3k+2} + \frac{1}{3k+4} > \frac{2}{3k+3},$$

i.e.,

$$\frac{2(3k+3)}{(3k+3)^2-1} > \frac{2}{3k+3},$$

i.e.,

$$\frac{1}{(3k+3)^2-1} > \frac{1}{(3k+3)^2},$$

which holds for all $k \geqq 1$.

Since, in addition, $f(1) = 13/12 > 1$, we conclude that $f(n) > 1$ for all natural numbers n.

4.5 Starting with the inequalities

$$\frac{1}{r(r+1)} < \frac{1}{r^2} < \frac{1}{r^2-1} \quad (r > 1),$$

prove that

$$\frac{m}{(m+1)(2m+1)} < \sum_{r=m+1}^{2m} \frac{1}{r^2} < \frac{m}{(m+1)(2m+1)} + \frac{3m+1}{4m(m+1)(2m+1)}$$

for m a natural number.

4.6 Show that if $a_1 \geqq a_2 \geqq a_3 \geqq \ldots \geqq a_n \geqq 0$ and

$$\sum_{\nu=1}^{k} a_\nu \leqq \sum_{\nu=1}^{k} b_\nu \quad (k = 1, 2, \ldots, n), \tag{1}$$

then

$$\sum_{\nu=1}^{n} a_\nu^2 \leqq \sum_{\nu=1}^{n} b_\nu^2. \tag{2}$$

PROOF. After multiplication by $a_k - a_{k+1}$, (1) becomes

$$(a_k - a_{k+1}) \sum_{\nu=1}^{k} a_\nu \leqq (a_k - a_{k+1}) \sum_{\nu=1}^{k} b_\nu \quad (k = 1, 2, \ldots, n), \tag{3}$$

where $a_{n+1} = 0$. Summing both sides of the relation (3) from $k = 1$ to $k = n$, we obtain

$$\sum_{\nu=1}^{n} a_\nu^2 \leqq \sum_{\nu=1}^{n} a_\nu b_\nu. \tag{4}$$

With Schwarz's inequality

$$\left(\sum_{\nu=1}^{n} a_\nu b_\nu\right)^2 \leqq \left(\sum_{\nu=1}^{n} a_\nu^2\right) \left(\sum_{\nu=1}^{n} b_\nu^2\right),$$

the relation (4) yields

$$\left(\sum_{\nu=1}^{n} a_\nu^2\right)^2 \leqq \left(\sum_{\nu=1}^{n} a_\nu b_\nu\right)^2 \leqq \left(\sum_{\nu=1}^{n} a_\nu^2\right) \left(\sum_{\nu=1}^{n} b_\nu^2\right),$$

whence follows (2).

Equality holds in (2) if and only if $a_\nu = b_\nu$ $(\nu = 1, 2, \ldots, n)$.

4.7 If $x_i \leqq x_k$ $(i < k)$, $y_i \leqq y_k$ $(i < k)$ or if $x_i \geqq x_k$ $(i < k)$, $y_i \geqq y_k$ $(i < k)$, then prove that $D_n \leqq D_{n+1}$, where

$$D_r = r \sum_{\nu=1}^{r} x_\nu y_\nu - \left(\sum_{\nu=1}^{r} x_\nu\right) \left(\sum_{\nu=1}^{r} y_\nu\right).$$

4.8 Prove the inequality

$$a+a^2+\dots+a^{2n} \leqq n(a^{2n+1}+1) \quad (a \geqq 0, n \text{ a natural number}). \quad (1)$$

SOLUTION. For $n = 1$, the relation (1) becomes

$$a+a^2 \leqq a^3+1 (a \geqq 0) \Leftrightarrow a(a+1) \leqq (a^2-a+1)(a+1), \quad (2)$$

which is valid, because

$$a \leqq (a-1)^2+a \text{ for all } a.$$

Equality holds for $a = 1$.

Let us now suppose that (1) is valid for $n = k$, i.e., that

$$f(k) \equiv a+a^2+\dots+a^{2k} \leqq k(a^{2k+1}+1) \equiv g(k) \quad (a \geqq 0), \quad (3)$$

which may be more briefly written as

$$f(k) \leqq g(k) \quad (a \geqq 0). \quad (4)$$

Then

$$f(k+1) \leqq k(a^{2k+1}+1)+a^{2k+1}+a^{2k+2}.$$

Consequently, if

$$k(a^{2k+1}+1)+a^{2k+1}+a^{2k+2} \leqq (k+1)(a^{2k+3}+1) \quad (a \geqq 0), \quad (5)$$

then

$$f(k+1) \leqq g(k+1) \quad (a \geqq 0),$$

whenever (4) holds.

The inequality (5) is equivalent to

$$(k+1)a^{2k+1}(1-a^2) \leqq 1-(a^2)^{k+1} \quad (a \geqq 0). \quad (6)$$

We will prove that (6) is valid for $a \geqq 0$. If $0 \leqq a < 1$, after division by $1-a^2$, (6) is equivalent to

$$(k+1)a^{2k+1} \leqq 1+a^2+\dots+a^{2k}. \quad (7)$$

For $0 \leqq a < 1$, we have the inequalities

$$a^{2k+1} < 1,\ a^{2k+1} < a^2, \dots, a^{2k+1} < a^{2k},$$

from which follows the inequality (7). For $a = 1$, equality holds in (7).

If $a > 1$, then (6), after division by $1-a^2$, becomes

$$(k+1)a^{2k+1} \geqq 1+a^2+\ldots+a^{2k}. \tag{8}$$

Since $a > 1$, we have the inequalities

$$a^{2k+1} > 1,\ a^{2k+1} > a^2,\ \ldots,\ a^{2k+1} > a^{2k},$$

whence follows (8).

Thus (5) holds for all $a \geqq 0$. This establishes the inequality (1).

4.9 Prove the inequality

$$\sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k} \leqq \frac{n}{4} \tag{1}$$

(n a natural number, x a real number).

Proof. We start with the binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n. \tag{2}$$

After differentiating with respect to t and multiplying by t, we obtain

$$\sum_{k=0}^{n} k\binom{n}{k} t^k = nt(1+t)^{n-1}. \tag{3}$$

After another differentiation with respect to t and multiplication by t, this yields

$$\begin{aligned}\sum_{k=0}^{n} k^2 \binom{n}{k} t^k &= nt(1+t)^{n-1}+n(n-1)t^2(1+t)^{n-2}\\ &= nt(1+nt)(1+t)^{n-2}.\end{aligned} \tag{4}$$

If $x \neq 1$, the relations (2), (3), (4) with $t = x/(1-x)$ become, respectively,

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = 1, \tag{5}$$

$$\sum_{k=0}^{n} k \binom{n}{k} x^k(1-x)^{n-k} = nx, \tag{6}$$

$$\sum_{k=0}^{n} k^2 \binom{n}{k} x^k(1-x)^{n-k} = nx(1-x+nx). \tag{7}$$

Direct verification shows that (5), (6), and (7) also hold for $x = 1$.

If we multiply both sides of (5) by n^2x^2, (6) by $(-2nx)$ and (7) by 1, and add, we obtain

$$\sum_{k=0}^{n} (n^2x^2-2nxk+k^2)x^k\binom{n}{k}(1-x)^{n-k} = nx(1-x+nx)-n^2x^2,$$

i.e.,

$$\sum_{k=0}^{n} (nx-k)^2x^k\binom{n}{k}(1-x)^{n-k} = nx(1-x). \tag{8}$$

Next, since

$$x(1-x) = \tfrac{1}{4}-(x-\tfrac{1}{2})^2,$$

we have

$$x(1-x) \leqq 1/4. \tag{9}$$

From (8) and (9), the inequality (1) follows.

4.10 Prove that

$$\sum_{k=1}^{n^2}\left[\frac{n}{\sqrt{k}}\right]-\sum_{k=1}^{n}\left[\frac{n}{k}\right] \geqq n^2-n \quad (n \text{ a natural number}),$$

where $[a]$ represents the greatest integer not exceeding a.

4.11 From the graph of the function $x^p (p > 0)$ deduce the inequality

$$a^p < \frac{1}{b-a}\int_a^b x^p\,dx < b^p \quad (0 < a < b)$$

and hence prove the inequality

$$\frac{n^{p+1}}{p+1} < 1^p+2^p+3^p+\ldots+n^p < \frac{(n+1)^{p+1}-1}{p+1}.$$

4.12 Show that if $a_i \geqq -1$ $(i = 1, 2, \ldots, m)$,

$$b_k \leqq 1 \ (k = 1, 2, \ldots, n),$$

and

$$\sum_{i=1}^{m} a_i \leqq \sum_{k=1}^{n} b_k,$$

then

$$\left\{\prod_{i=1}^{m}(1+a_i)\right\}\left\{\prod_{k=1}^{n}(1-b_k)\right\} \leqq 1.$$

4.13 Prove the inequality

$$\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{n^2}>1 \quad (n \text{ a natural number} > 1)$$

PROOF. Since, for $n > 1$,

$$\frac{1}{n+1}>\frac{1}{n^2},\ \frac{1}{n+2}>\frac{1}{n^2},\ \ldots,\ \frac{1}{n^2-1}>\frac{1}{n^2},$$

we find

$$\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{n^2}>\frac{1}{n}+\frac{n^2-n}{n^2}=1.$$

4.14 Prove the inequality

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n} \qquad (n \geqq 2).$$

PROOF. If $k < n$ $(\geqq 2)$, then $1/\sqrt{k} > 1/\sqrt{n}$. Setting $k = 1$ 2, ..., n, in succession, and adding the inequalities so obtained we find

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>n\frac{1}{\sqrt{n}}=\sqrt{n},$$

which is the desired inequality.
(Solution by B. Mesihović.)

4.15 $\left(\sum_{k=1}^{n} k^2\right)^{1/2} \geqq \left(\sum_{k=1}^{n} k^3\right)^{1/3}.$

4.16 $\prod_{k=1}^{n}(1+a_k) > 1+\sum_{k=1}^{n} a_k \quad (a_k > 0 \text{ or } -1 < a_k < 0;\ n > 1)$

4.17 $\prod_{k=1}^{n}(1-a_k) > 1-\sum_{k=1}^{n} a_k \quad (0 < a_k < 1;\ n > 1).$

4.18 $\prod_{k=1}^{n}(1+a_k) < 1\Big/\left(1-\sum_{k=1}^{n} a_k\right) \quad \left(\sum_{k=1}^{n} a_k < 1,\ a_k > 0\right).$

4.19 $\prod_{k=1}^{n}(1-a_k) < 1\Big/\left(1+\sum_{k=1}^{n} a_k\right) \quad (1 > a_k > 0).$

4.20 $\prod_{k=1}^{n}(1+a_k) \leqq 1+s_n+\frac{s_n^2}{2!}+\ldots+\frac{s_n^n}{n!} \quad \left(a_k \geqq 0,\ s_n = \sum_{k=1}^{n} a_k\right).$

4.21 $\prod_{k=0}^{n}(1-\tfrac{1}{2}a^{2^k}) \geqq 1-a+\frac{a}{2^{n+1}} \qquad (0 \leqq a \leqq 1).$

4.22 $\prod_{k=1}^{n-1}(1+a^k) < (1-a)/(1-2a+a^n) \qquad (0 < a < \tfrac{1}{2}).$

4.23 $\prod_{k=1}^{n}(1+a_k) \geqq \frac{2^n}{n+1}\left(1+\sum_{k=1}^{n} a_k\right) \qquad (a_k \geqq 1).$

4.24 $\exp\left(\sum_{k=1}^{n}\frac{1}{k}\right) > n+1.$

4.25 $\exp\left(\sum_{k=1}^{n}\frac{1}{k}\right) > n+\frac{17}{10}.$

4.26 $\exp\left(\sum_{k=1}^{n}\frac{1}{k}\right) > \frac{n+1}{2}e \qquad (n > 1).$

4.27 $\tfrac{1}{2}n^2 < \sum_{k=1}^{n} k < \tfrac{1}{2}(n+1)^2.$

4.28 $\tfrac{1}{3}n^3 < \sum_{k=1}^{n} k^2 < \tfrac{1}{3}(n+1)^3.$

4.29 $\tfrac{1}{2}-\frac{1}{n+1} < \sum_{k=2}^{n}\frac{1}{k^2} < 1-\frac{1}{n}.$

4.30 $\frac{1}{n+1}-\frac{1}{n+p+1} < \sum_{k=1}^{p}\frac{1}{(n+k)^2} < \frac{1}{n}-\frac{1}{n+p}.$

4.31 $\sum_{k=1}^{n}\frac{1}{k(k+1)} < 1.$

4.32 $\sum_{k=1}^{n} \frac{1}{k^2} < 2.$

4.33 $\frac{1}{2} < \sum_{k=1}^{n} \frac{1}{n+k} < \frac{\sqrt{2}}{2} \quad (n > 1).$

4.34 $\frac{n}{2} < \sum_{k=1}^{2^n-1} \frac{1}{k} < n \quad (n > 1).$

4.35 $n\{(n+1)^{1/n}-1\} < \sum_{k=1}^{n} \frac{1}{k} < n\{1+(n+1)^{-1}-(n+1)^{-1/n}\}.$

4.36 $\sum_{k=0}^{2n} \frac{x^k}{k!} > 0.$

4.37 $\sum_{k=0}^{n-1} \frac{n-k}{m-k} < \frac{n(n+1)}{2m-n+1} \qquad (m > n > 1).$

4.38 $2n+1 < \sum_{k=1}^{2n+1} k \frac{2n-k+2}{2n+1} < (n+1)^2.$

4.39 $\frac{1}{p} \sum_{\nu=1}^{p} \nu^n \geqq \left(\frac{p+1}{2}\right)^n.$

4.40 $\sum_{k=2}^{n} \frac{1}{k} < \log n.$

4.41 $(2n)!\left(\sum_{k=2}^{2n} \frac{(-1)^k}{k!}\right) \geqq \{(2n-1)!!\}^2.$

4.42 $\sum_{k=1}^{n} \frac{(n!)^4}{\{(n-k)!\}^2\{(n+k)!\}^2} \leqq \frac{\sqrt{\pi}}{2\sqrt{2}}.$

4.43 $\frac{1}{n} + \frac{2}{n-1} + \frac{3}{n-2} + \dots + \frac{n}{1}$

$$> \frac{n+1}{2n}\left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{1}\right) \quad (n \geqq 2)$$

§ 5. Inequalities Involving Trigonometric Functions

5.1 Prove the inequality

$$\left|\sin\left(\sum_{k=1}^{\nu} x_k\right)\right| < \sum_{k=1}^{\nu} \sin x_k \ (0 < x_k < \pi;\ k = 1, 2, \ldots;\ \nu > 1). \qquad (1)$$

SOLUTION. First of all, we have

$$\begin{aligned}\sin(x_1+x_2) &= \sin x_1 \cos x_2 + \cos x_1 \sin x_2\\ &\Rightarrow |\sin(x_1+x_2)| < |\sin x_1| + |\sin x_2|. \qquad (2)\end{aligned}$$

Since, by hypothesis,

$$0 < x_1 < \pi,\ 0 < x_2 < \pi,$$

it follows from (2) that

$$|\sin(x_1+x_2)| < \sin x_1 + \sin x_2.$$

Accordingly, the inequality (1) holds for $\nu = 2$. Let us suppose that it is valid for some $n (\geqq 2)$ and consider the equality

$$\sin\left(x_{n+1} + \sum_{k=1}^{n} x_k\right) = \sin x_{n+1} \cos\left(\sum_{k=1}^{n} x_k\right) + \cos x_{n+1} \sin\left(\sum_{k=1}^{n} x_k\right).$$

Hence

$$\begin{aligned}\left|\sin\left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\right| &< |\sin x_{n+1}| + \left|\sin\left(\sum_{k=1}^{n} x_k\right)\right|\\ &< |\sin x_{n+1}| + \sum_{k=1}^{n} \sin x_k\end{aligned}$$

by the induction hypothesis.

Since $0 < x_{n+1} < \pi$, the preceding inequality gives

$$\left|\sin\left(\sum_{k=1}^{n+1} x_k\right)\right| < \sum_{k=1}^{n+1} \sin x_k \qquad (0 < x_k < \pi;\ k = 1, 2, \ldots;\ n > 1).$$

This completes the inductive proof.

5.2 Compute the sums

$$S_1 = \sum_{k=n+1}^{m} \cos kx,\ S_2 = \sum_{k=n+1}^{m} \sin kx \qquad (m \text{ a natural number} > n).$$

Prove that

$$(S_1^2+S_2^2)^{1/2} \leqq 1/(\sin \tfrac{1}{2}x) \quad (0 < x < 2\pi). \tag{1}$$

ANSWER.

$$S_1 = \{\cos \tfrac{1}{2}(m+n+1)x \sin \tfrac{1}{2}(m-n)x\}/(\sin \tfrac{1}{2}x),$$
$$S_2 = \{\sin \tfrac{1}{2}(m+n+1)x \sin \tfrac{1}{2}(m-n)x\}/(\sin \tfrac{1}{2}x).$$

NOTE. How does the inequality (1) change if $x \in (-\infty, +\infty)$?

5.3 Solve the inequality $\sin x > \sin 3x$ graphically.

5.4 If $0 < a_k < \pi$ $(k = 1, 2, \ldots, n)$, is it true that

$$\prod_{k=1}^{n} \sin a_k \leqq \left\{\sin \frac{1}{n} \sum_{k=1}^{n} a_k\right\}^n ?$$

5.5 Prove the inequality

$$\sum_{i<j}^{n} \cos(a_i - a_j) \geqq -\tfrac{1}{2}n \quad (a_1, a_2, \ldots, a_n \text{ real}). \tag{1}$$

PROOF.

$$\left(\sum_{i=1}^{n} \cos a_i\right)^2 + \left(\sum_{i=1}^{n} \sin a_i\right)^2$$
$$= n + 2\sum_{i<j}^{n} (\cos a_i \cos a_j + \sin a_i \sin a_j)$$
$$= n + 2\sum_{i<j}^{n} \cos(a_i - a_j),$$

whence the inequality (1) follows directly.

5.6 For what values of x is

$$\sin x > 2\cos^2 x - 1?$$

SOLUTION. The given inequality may be written in the form

$$2\sin^2 x + \sin x - 1 > 0$$

or

$$2(\sin x + 1)(\sin x - \tfrac{1}{2}) > 0.$$

Since $\sin x > -1$ for all $x \neq 3\pi/2+2k\pi$, the above inequality is satisfied provided that

$$\sin x-\tfrac{1}{2} > 0 \Leftrightarrow x \in (\pi/6+2k\pi,\ 5\pi/6+2k\pi) \quad (k = 0, \pm 1, \pm 2, \ldots).$$

5.7 Show that if n is a natural number, then

$$\left|\frac{\sin nx}{\sin x}\right| \leqq n \quad (x \neq k\pi,\ k = 0,\ \pm 1,\ \pm 2, \ldots). \tag{1}$$

SOLUTION 1. For $n = 1$, the relation is valid. Let us suppose that (1) holds for $n = k$, i.e., that

$$\left|\frac{\sin kx}{\sin x}\right| \leqq k. \tag{2}$$

Consider the quotient

$$\frac{\sin(k+1)x}{\sin x} = \frac{\sin kx}{\sin x}\cos x+\cos kx.$$

It follows from this result that

$$\left|\frac{\sin(k+1)x}{\sin x}\right| \leqq \left|\frac{\sin kx}{\sin x}\right| |\cos x|+|\cos kx|.$$

From (2) and the preceding relation, it follows that

$$\left|\frac{\sin(k+1)x}{\sin x}\right| \leqq k+1.$$

This proves (1) by induction.

SOLUTION 2. The inequality (1) may also be proved by starting with the identities

$$\frac{\sin 2px}{\sin x} = 2\{\cos x+\cos 3x+ \ldots +\cos(2p-1)x\} \quad (p \geqq 1)$$

and

$$\frac{\sin(2p+1)x}{\sin x} = 1+2\{\cos 2x+\cos 4x+ \ldots +\cos 2px\} \quad (p \geqq 1).$$

From these it follows that

$$\left|\frac{\sin 2px}{\sin x}\right| \leqq 2p, \left|\frac{\sin(2p+1)x}{\sin x}\right| \leqq 2p+1.$$

5.8 Show that, if k is a natural number and if A and B are real numbers, then

$$\left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| \leqq k^2 \ (\cos A \neq \cos B). \tag{1}$$

PROOF. Starting with the identity

$$\cos p - \cos q = -2 \sin \tfrac{1}{2}(p+q) \sin \tfrac{1}{2}(p-q),$$

we find that

$$\frac{\cos kB - \cos kA}{\cos B - \cos A} = \frac{\sin \frac{1}{2}k(B+A) \sin \frac{1}{2}k(B-A)}{\sin \frac{1}{2}(B+A) \sin \frac{1}{2}(B-A)}$$

$$\Rightarrow \left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| = \left|\frac{\sin \frac{1}{2}k(B+A)}{\sin \frac{1}{2}(B+A)}\right| \left|\frac{\sin \frac{1}{2}k(B-A)}{\sin \frac{1}{2}(B-A)}\right|.$$

Since

$$|\sin nx/\sin x| \leqq n \ (n \text{ a natural number}), \tag{2}$$

the preceding inequality gives precisely the inequality (1).

The inequality (2) was proved in **5.7**.

5.9 If A and B ($\cos B \neq \cos A$) are real numbers and if $k(>1)$ is a natural number, then

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| < k^2 - 1. \tag{1}$$

SOLUTION. From the inequality

$$|\sin rx| \leqq r\,|\sin x| \ (r \text{ a natural number})$$

it follows that

$$\left|\frac{\sin rx \sin sy}{\sin x \sin y}\right| \leqq rs \Rightarrow \left|\frac{\sin rx \sin sy + \sin sx \sin ry}{\sin x \sin y}\right| \leqq 2rs$$

(r, s natural numbers). If we now set $r = k+1$, $s = k-1$, $x = \frac{1}{2}(A+B)$, $y = \frac{1}{2}(A-B)$, we obtain the inequality (1).

This solution was given by L. E. Bush (*The American Mathematical Monthly*, vol. 64, 1957, p. 651).

5.10 Is there an a such that

$$1-\frac{t^2}{2}+\frac{t^4}{24} > \cos t > 1-\frac{t^2}{2}+\frac{t^4}{a} \quad (0 < t < \pi/2)?$$

5.11 What conditions must be satisfied by the positive numbers a and b so that the following inequality is valid:

$$x \geqq \frac{(a+b)\sin x}{a+b\cos x} \quad (x \geqq 0)? \tag{1}$$

SOLUTION. If we set $b/a = k$ (> 0), the inequality (1) becomes

$$x \geqq \frac{(1+k)\sin x}{1+k\cos x} \quad (x \geqq 0). \tag{2}$$

We cannot have $k > 1$, because when x tends to a root of the equality

$$1+k\cos x = 0,$$

(and such roots will exist), then the value of the function

$$\frac{(1+k)\sin x}{1+k\cos x}$$

tends to infinity and (2) cannot hold.

For $k = 1$, the inequality (2) becomes

$$x \geqq \frac{2\sin x}{1+\cos x} = 2\tan \tfrac{1}{2}x \quad (x \geqq 0),$$

which is false. Consequently, we must take $k < 1$. Then (2) may be written in the form:

$$x-\sin x+k(x\cos x-\sin x) \geqq 0,$$

$$f(x) = \frac{x\cos x-\sin x}{x-\sin x} \geqq -\frac{1}{k} \quad (x \geqq 0).$$

Here we take $f(0) = \lim_{x\to 0} f(x)$.

If m denotes the minimum value of the function $f(x)$, we seek the conditions under which

$$m \geqq -1/k = -a/b,$$

i.e.,

$$a+mb \geqq 0. \tag{3}$$

We shall show that $m = f(0)$. First of all, we use L'Hôpital's method to find

$$\begin{aligned} f(0) &= \lim_{x\to 0} \frac{x \cos x - \sin x}{x - \sin x} = \lim_{x\to 0} \frac{-x \sin x}{1 - \cos x} \\ &= \lim_{x\to 0} \frac{-\sin x - x \cos x}{\sin x} = \lim_{x\to 0} \frac{x \sin x - 2 \cos x}{\cos x} \\ &= -2. \end{aligned}$$

We must prove that $f(x) \geqq -2$, i.e., that

$$x \cos x - \sin x \geqq -2x + 2 \sin x \quad (x \geqq 0),$$

$$x(2+ \cos x) \geqq 3 \sin x \quad (x \geqq 0),$$

$$g(x) = x - \frac{3 \sin x}{2+ \cos x} \geqq 0 \quad (x \geqq 0). \tag{4}$$

Since

$$g'(x) = 1 - 3\frac{(2+\cos x)\cos x + \sin^2 x}{(2+\cos x)^2} = \left(\frac{1-\cos x}{2+\cos x}\right)^2 \geqq 0,$$

the function $g(x)$ is increasing, so that

$$g(x) \geqq g(0) = 0 \quad (x \geqq 0),$$

which is identical with the inequality (4). Hence $m = -2$, and the condition (3) may be written as $a \geqq 2b$. Thus, we have proved the inequality

$$x \geqq \frac{(a+b)\sin x}{a+b \cos x} \quad (a \geqq 2b > 0, x \geqq 0).$$

(Solution by D. Djoković.)

5.12 Prove Everitt's inequalities:

$$\left.\begin{aligned}
&\frac{\sin\theta}{\theta} < \frac{2(1-\cos\theta)}{\theta^2},\\
&\frac{\sin\theta}{\theta} < \frac{2+\cos\theta}{3},\\
&\frac{\sin\theta}{\theta} > \frac{4-4\cos\theta-\theta^2}{\theta^2},\\
&\frac{\sin\theta}{\theta} > \frac{\cos\theta+4-\theta^2/3}{5}.
\end{aligned}\right\} \quad (0 < \theta \leqq \pi).$$

NOTE. Concerning these inequalities and their generalizations see: *The Mathematical Gazette*, vol. 44, 1960, p. 52—54.

5.13 Prove the inequality

$$\sqrt{A}+\sqrt{B}+\sqrt{C} \leqq 4\sqrt{3},$$

where

$$A = \tan\beta\tan\gamma+5,\ B = \tan\gamma\tan\alpha+5,\ C = \tan\alpha\tan\beta+5,$$

$$\alpha > 0, \beta > 0, \gamma > 0, \alpha+\beta+\gamma = \tfrac{1}{2}\pi.$$

5.14 If $0 < x, y < \pi/2$ and $0 < x+y < \pi/2$, prove that

$$0 < \tan x \tan y < 1,\ \tan(x+y) > \tan x+\tan y.$$

5.15 Prove that

$$\sin\theta+\tfrac{1}{2}\sin 2\theta > 0 \quad (0 < \theta < \pi), \tag{1}$$

$$\sin\theta+\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta > 0 \quad (0 < \theta < \pi). \tag{2}$$

PROOF. From the identity

$$\sin\theta+\tfrac{1}{2}\sin 2\theta \equiv \sin\theta(1+\cos\theta),$$

since

$$\sin\theta > 0 \text{ and } 1+\cos\theta > 0 \quad (0 < \theta < \pi),$$

(1) follows immediately.

In order to prove (2), we consider the identity

$$\sin\theta+\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta \equiv \sin\theta+\sin\theta\cos\theta+\tfrac{1}{3}\sin\theta\,(4\cos^2\theta-1)$$

$$\equiv \tfrac{1}{3}\sin\theta(2+3\cos\theta+4\cos^2\theta)$$
$$\equiv \tfrac{1}{3}\sin\theta\{(1+\cos\theta)^2+(1+\cos\theta)+3\cos^2\theta\},$$

from which (2) follows at once.

The inequalities (1) and (2) are particular cases of Jackson's inequality (see § 0.8).

5.16 Prove that

$$\sin\theta+\sin 2\theta+\ldots+\sin n\theta+\tfrac{1}{2}\sin(n+1)\theta \geqq 0 \quad (0 \leqq \theta \leqq \pi). \quad (1)$$

PROOF. The left-hand side of (1) may be written in the form

$$\tfrac{1}{2}\left(\sum_{k=1}^{n}\sin k\theta+\sum_{k=1}^{n+1}\sin k\theta\right). \quad (2)$$

Since

$$\sum_{k=1}^{n}\sin k\theta \equiv \frac{\sin\frac{n}{2}\theta\,\sin\frac{n+1}{2}\theta}{\sin\frac{1}{2}\theta} \quad (0<\theta<\pi),$$

the expression (2) is equal to

$$\frac{1}{2}\left(\frac{\sin\frac{n}{2}\theta\,\sin\frac{n+1}{2}\theta}{\sin\frac{1}{2}\theta}+\frac{\sin\frac{n+1}{2}\theta\,\sin\frac{n+2}{2}\theta}{\sin\frac{1}{2}\theta}\right)$$

$$\equiv \frac{1}{2}\,\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{1}{2}\theta}\left(\sin\frac{n}{2}\theta+\sin\frac{n+2}{2}\theta\right)$$

$$\equiv \sin^2\frac{n+1}{2}\theta\cot\tfrac{1}{2}\theta.$$

This expression is non-negative for all values of $\theta \in (0, \pi)$. Finally, (1) is trivially true if $\theta = 0$ or π.

5.17 Prove that

$$\operatorname{arc}\cos x \geqq \sqrt{1-x^2} \quad (-1 \leqq x \leqq +1).^*$$

*) This assumes the principal-value branch of the inverse function.

PROOF. For $t \geqq 0$,

$$t \geqq \sin t = \sqrt{1-\cos^2 t}.$$

If $x \in [-1, +1]$, we may set $t = \text{arc}\cos x$ i.e., $x = \cos t$. Consequently,

$$\text{arc}\cos x \geqq \sqrt{1-x^2} \quad (-1 \leqq x \leqq +1).$$

5.18 Prove that

$$\text{arc}\sin x < x+\sqrt{x} \quad (0 < x \leqq 1).$$

PROOF. If we set $x = \sin t$ $(0 < t \leqq \pi/2)$, the given inequality becomes

$$t-\sin t < \sqrt{\sin t}. \tag{1}$$

By Taylor's formula,

$$\sin t = t-\frac{t^3}{6}\cos \theta t \quad (0 < \theta < 1),$$

i.e.,

$$t-\sin t = \frac{t^3}{6}\cos \theta t < \frac{t^3}{6}.$$

The inequality (1) follows from the inequality

$$\frac{t^3}{6} < \sqrt{\sin t},$$

i.e.,

$$\frac{t^6}{36} < \sin t \quad (0 < t \leqq \pi/2), \tag{2}$$

which follows from Jordan's inequality

$$\frac{2t}{\pi} \leqq \sin t \quad (0 < t \leqq \pi/2),$$

because in the interval under consideration

$$\frac{t^6}{36} < \frac{2t}{\pi}, \text{ i.e., } t^5 < 72/\pi.$$

5.19 Prove the inequality

$$1-2\theta/\pi \leqq \cos\theta \leqq \pi/2-\theta \quad (0 \leqq \theta \leqq \pi/2).$$

HINT. Use Jordan's inequality (§ 0.9).

5.20 $\tan a_1 < \left(\sum_{k=1}^{n} \sin a_k\right) \Big/ \left(\sum_{k=1}^{n} \cos a_k\right) < \tan a_n$

$$(0 < a_1 < a_2 < \ldots < a_n < \tfrac{1}{2}\pi).$$

5.21 Prove the inequality

$$2(\cos a-\sin a)^2 \leqq [\cos(a+\theta)+\sin(a+\theta)]^2 + [\cos(a-\theta)+\sin(a-\theta)]^2 \leqq 2(\cos a+\sin a)^2$$

where $0 < a < \pi/2$ and θ is arbitrary.

SOLUTION. Taking into account that

$$[\cos(a+\theta)+\sin(a+\theta)]^2+[\cos(a-\theta)+\sin(a-\theta)]^2$$
$$= 2+\sin 2(a+\theta)+\sin 2(a-\theta) = 2+2\sin 2a\cos 2\theta,$$
$$2(\cos a-\sin a)^2 = 2-2\sin 2a,$$
$$2(\cos a+\sin a)^2 = 2+2\sin 2a,$$

the given inequality reduces to

$$-1 \leqq \cos 2\theta \leqq +1.$$

5.22 $\sin x+\tan x > 2x \quad (0 < x < \pi/2).$

5.23 $1 \geqq \cos x \geqq 1-\frac{1}{2}x^2.$

5.24 $\tan x > x+\frac{1}{3}x^3 \quad (0 < x < \pi/2).$

5.25 $\arctan x < x-\frac{1}{6}x^3 \quad (0 < x \leqq 1).$

5.26 $x-\frac{1}{3}x^3 < \arctan x \quad (x > 0).$

5.27 $-\log\cos x < \frac{1}{2}\sin x\tan x \quad (0 < x < \pi/2).$

5.28 $(b-a)\cos b \leqq \sin b-\sin a \leqq (b-a)\cos a \quad (0 < a, b < \pi/2).$

5.29 $\pi < \dfrac{\sin \pi x}{x(1-x)} \leqq 4 \quad (0 < x < 1).$

5.30 $\dfrac{\tan a}{a} < \dfrac{\tan b}{b}, \dfrac{\sin a}{a} > \dfrac{\sin b}{b} \quad (a < b;\ 0 < a,\ b < \tfrac{1}{2}\pi).$

5.31 $\left| \dfrac{d^n}{dx^n} \left(\dfrac{\sin x}{x} \right) \right| \leqq \dfrac{1}{n+1}.$

5.32 $\left| \dfrac{d^n}{dx^n} \left(\dfrac{1-\cos x}{x} \right) \right| \leqq \dfrac{1}{n+1}.$

5.33 $\cos x + x \sin x > 1 \quad (0 < x \leqq \pi/2).$

5.34 $\tan x < x + x^3 \quad (0 < x \leqq \pi/3).$

5.35 $-\sqrt{3} \leqq \dfrac{3 \sin x}{2+\cos x} \leqq \sqrt{3}.$

§ 6. Geometric Inequalities

6.1 Show that if α, β, γ are the angles of a triangle ABC, then $\cos \alpha + \cos \beta + \cos \gamma \leqq 3/2$.

Solution. Let $|\overrightarrow{BC}| = a$, $|\overrightarrow{CA}| = b$, $|\overrightarrow{AB}| = c$. Since

$\overrightarrow{AB} \cdot \overrightarrow{BC} = -ca \cos \beta$, $\overrightarrow{BC} \cdot \overrightarrow{CA} = -ab \cos \gamma$, $\overrightarrow{CA} \cdot \overrightarrow{AB} = -bc \cos \alpha$,

we find

$$0 \leqq \left(\frac{\overrightarrow{BC}}{a} + \frac{\overrightarrow{CA}}{b} + \frac{\overrightarrow{AB}}{c} \right)^2 = 3 - 2(\cos \alpha + \cos \beta + \cos \gamma),$$

whence

$$\cos \alpha + \cos \beta + \cos \gamma \leqq 3/2.$$

(Solution by D. C. B. Marsh.)

6.2 Show that if A, B, C are the angles of a triangle, then

1°: $(\sin A/2 + \sin B/2 + \sin C/2)^2 \leqq \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2;$

2°: $\cos A/2 \cos B/2 \cos C/2 \leqq \dfrac{3\sqrt{3}}{8};$

3°: $\tan^2 A/2 + \tan^2 B/2 + \tan^2 C/2 \geqq 1.$

6.3 Let a, b, c be the sides of a triangle ABC of area S.

Prove the inequality

$$4S\sqrt{3} \leqq a^2+b^2+c^2. \tag{1}$$

SOLUTION. Let us suppose that (1) is false, so that

$$4S\sqrt{3} > a^2+b^2+c^2. \tag{2}$$

The inequality (2) is equivalent to

$$2bc \sin \alpha > \frac{1}{\sqrt{3}} (a^2+b^2+c^2), \tag{3}$$

where α is the angle opposite the side a. By the law of cosines,

$$2bc \cos \alpha = b^2+c^2-a^2. \tag{4}$$

Squaring (3) and (4) and adding, we obtain the inequality

$$a^2b^2+b^2c^2+c^2a^2 > a^4+b^4+c^4,$$

which is equivalent to

$$(a^2-b^2)^2+(a^2-c^2)^2+(b^2-c^2)^2 < 0,$$

which is false.

From this result it follows that the inequality (1) is valid.

6.4 Let P be any point in the interior of the triangle ABC and let $|PA| = x$, $|PB| = y$, $|PC| = z$. Let p, q, r be the distances of P from the sides BC, CA, AB, respectively.

Prove the inequality

$$xyz \geqq (q+r)(r+p)(q+p).$$

6.5 With the notation of **6.4**, prove that

$$x+y+z \geqq 2(p+q+r).$$

SOLUTION. If L, M, N are the feet of the perpendicular from P to BC, CA, AB, respectively, we have

$$MN = (q^2+r^2+2qr \cos \alpha)^{\frac{1}{2}},\ MN = x \sin \alpha,$$

where α, β, γ are the angles of the triangle. We now have, in turn,

$$x+y+z = \sum_{\alpha,\beta,\gamma} (q^2+r^2+2qr\cos\alpha)^{1/2}/\sin a$$

$$= \sum [(q\sin\gamma+r\sin\beta)^2+(q\cos\gamma-r\cos\beta)^2]^{1/2}/\sin\alpha$$

$$\geqq \sum (q\sin\gamma+r\sin\beta)/\sin\alpha = \sum p\left(\frac{\sin\beta}{\sin\gamma}+\frac{\sin\gamma}{\sin\beta}\right)$$

$$\geqq 2(p+q+r).$$

(Solution by L. J. Mordell.)

6.6 With the notation of **6.4**, prove that

$$px+qy+rz \geqq 2(qr+rp+pq), \tag{1}$$

$$xyz \geqq 8pqr. \tag{2}$$

SOLUTION. Denote the sides of the triangle by a, b, c and its area by Δ. If h_a is the altitude from A,

$$a(x+p) \geqq ah_a = 2\Delta = ap+bq+cr,$$

whence

$$ax \geqq bq+cr. \tag{3}$$

From (3)

$$px \geqq \left(\frac{b}{a}\right)pq+\left(\frac{c}{a}\right)pr,$$

whence

$$\sum px \geqq \left(\frac{b}{c}+\frac{c}{b}\right)qr \geqq 2\sum qr.$$

This proves (1). For (2), we apply the inequality of the arithmetic and geometric means to (3), obtaining

$$ax \geqq 2(bcqr)^{1/2}$$

and two similar inequalities. Multiplying these yields (2). (Note that (2) may also be deduced from the result of **6.4.**) (Problem and solution by A. Oppenheim.)

6.7 Show that a necessary and sufficient condition that line segments of lengths a, b, c can form a triangle is:

$$b^2c^2+c^2a^2+a^2b^2 > \tfrac{1}{2}(a^4+b^4+c^4).$$

6.8 Show that if a_1, a_2, a_3 are the sides of the triangle ABC and h_1, h_2, h_3 the corresponding altitudes, then

$$\sqrt{3}\sum_{k=1}^{3} a_k \geqq 2\sum_{k=1}^{3} h_k.$$

Equality holds if and only if ABC is equilateral.

6.9 Show that if A, B, C are the angles of a triangle ABC, then

$$\sin A/2 \sin B/2 \sin C/2 \leqq 1/8.$$

6.10 Show that if a, b, c are the sides of any triangle, then

$$(a+b-c)(b+c-a)(c+a-b) \leqq abc.$$

6.11 If h_a is the altitude to the side BC of the triangle ABC ($|BC| = a$, $|CA| = b$, $|AB| = c$), prove that

$$h_a \leqq \sqrt{s(s-a)} \quad \left(s = \frac{a+b+c}{2}\right).$$

6.12 Show that if r is the radius of the inscribed circle of a triangle and R the radius of the circumscribed circle, then

$$\frac{r}{R} \leqq \frac{1}{2}.$$

6.13 A given point is at distances $\sqrt{2}$, 2 and $\sqrt{3}-1$ from the vertices of a triangle. Find the maximum area of the triangle.

Solution. Label the given point P and the vertices A, B, C with $PA = 2$, $PB = \sqrt{2}$, $PC = \sqrt{3}-1$. Let x denote the length of the perpendicular from P to AB. Then the optimum configuration gives an area of

$$\tfrac{1}{2}(\sqrt{3}-1+x)\{(4-x^2)^{1/2}+(2-x^2)^{1/2}\} \quad (0 \leqq x \leqq \sqrt{2}).$$

This area has a maximum of $\frac{1}{2}(3+\sqrt{3})$, at $x = 1$.

(Solution by D. C. B. Marsh.)

6.14 Let A_k $(k = 1, 2, 3, 4)$ be the vertices of a square and P a point in the same plane. Prove the inequality

$$\sum_{k=1}^{4} |A_kP| \geqq (1+\sqrt{2}) \max_k |A_kP|+\min_k|A_kP|$$

and determine the points P for which equality holds.

6.15 If S_k is the sum of all the perpendiculars from the center to the sides of a regular polygon of k sides which is inscribed in a circle of radius r, prove that

$$S_{k+1}-S_k > r.$$

6.16 About an arbitrary circle there is circumscribed a regular polygon of n sides each of length a_n, and within this circle is inscribed a regular polygon of k sides each of length b_k. Prove the inequalities:

$$a_{n+1} < b_n \quad (n = 5, 6, \ldots),$$

$$a_{n+1} > b_n \quad (n = 3, 4),$$

$$2a_{n+1} < b_n+b_{n+1} \quad (n = 9, 10, \ldots),$$

$$2a_{n+1} > b_n+b_{n+1} \quad (n = 3, 4, \ldots, 8).$$

This exercise may be found in the journal: *Fiziko-matematičesko spisanie*, Sofija, v. 2, 1959, p. 239.

6.17 Given a circle C of circumference U. Let p_n be the perimeter of the regular n-gon inscribed in C and P_n the perimeter of the circumscribed regular n-gon $(n \geqq 3)$. Prove that:

1°: The sequence $\{p_n\}$ is monotone increasing, and the sequence $\{P_n\}$ is monotone decreasing;

2°: $\frac{2}{3} p_n+\frac{1}{3} P_n > U$.

SOLUTION. 1°: The side of a regular inscribed n-gon measures $2r \sin (\pi/n)$ and the side of a regular circumscribed n-gon is $2r \tan (\pi/n)$, where r is the radius of the circle C. Therefore

$$p_n = 2rn \sin(\pi/n) = 2\pi r(n/\pi)\sin (\pi/n), \tag{1}$$

$$P_n = 2rn \tan(\pi/n) = 2\pi r(n/\pi) \tan(\pi/n). \tag{2}$$

For $0 < x < \pi/2$, the function $p(x) = \dfrac{\sin x}{x}$ is decreasing and the function $P(x) = \dfrac{\tan x}{x}$ is increasing, because

$$p'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x) < 0 \quad (0 < x < \pi/2),$$

$$P'(x) = \frac{\dfrac{x}{\cos^2 x} - \tan x}{x^2} = \frac{x - \sin x \cos x}{x^2 \cos^2 x} = \frac{2x - \sin 2x}{2\, x^2 \cos^2 x} > 0$$
$$(0 < x < \pi/2).$$

Since $0 < \pi/n < \pi/2$ $(n = 3, 4, \ldots)$, it follows that

$$p_n = 2\pi r\, p(\pi/n) \uparrow, \quad P_n = 2\pi r\, P(\pi/n) \downarrow \quad (n = 3, 4, \ldots).$$

2°: The inequality which we wish to prove may be written, by means of (1) and (2), in the form

$$\frac{2r}{3} n \left(2 \sin \frac{\pi}{n} + \tan \frac{\pi}{n}\right) > 2\pi r \quad (n = 3, 4, \ldots),$$

or

$$2 \sin \frac{\pi}{n} + \tan \frac{\pi}{n} > 3 \frac{\pi}{n} \quad (n = 3, 4, \ldots). \tag{3}$$

Let us set $f(x) = 2 \sin x + \tan x - 3x$. Since

$$f(0) = 0, f'(x) = 2 \cos x + \sec^2 x - 3, f'(0) = 0,$$

$$f''(x) = -2 \sin x + \frac{2 \sin x}{\cos^3 x} = \frac{2 \sin x}{\cos^3 x}(1 - \cos^3 x) > 0$$
$$(0 < x < \tfrac{1}{2}\pi),$$

the function $f(x)$ is strictly monotone increasing on the interval $[0, \pi/2)$, i.e.,

$$2 \sin x + \tan x > 3x \quad (0 < x < \pi/2). \tag{4}$$

The inequality (3) follows immediately from (4).
(Solution by D. Adamović.)

6.18 Given a tetrahedron with vertices A, B, C, D, let P be any point in this tetrahedron. Let A', B', C', D' be the orthogonal projections of the points A, B, C, D onto the faces BCD, CDA DAB, ABC. Prove the relation

$$\sum |PA| \text{ (area } \triangle BCD) \geqq 3 \sum |PA'| \text{ (area } \triangle BCD),$$

where, for example,

$$\sum |PA| \text{ (area } \triangle BCD) = |PA| (\text{area } \triangle BCD) + |PB| (\text{area } \triangle CDA) + |PC| (\text{area } \triangle DAB) + |PD| (\text{area } \triangle ABC).$$

HINT. Let H_A, H_B, H_C, H_D be the altitudes of the tetrahedron dropped to the triangular bases BCD, CDA, DAB, ABC. Then,

$$H_A \leqq |PA| + |PA'|, H_B \leqq |PB| + |PB'|, \text{ etc.}$$

6.19 Let P be any point in the interior of an arbitrary tetrahedron. Let p_1, p_2, p_3, p_4 be the distances from this point to the vertices of the tetrahedron, and let q_1, q_2, q_3, q_4 be the distances from this point to the faces of the tetrahedron. Prove the inequality

$$p_1 + p_2 + p_3 + p_4 \geqq 3(q_1 + q_2 + q_3 + q_4).$$

(This problem was contributed by W. Gridasow.)

6.20 Let a, b, c be the sides of a triangle; prove that

$$a^2 + b^2 + c^2 \geqq \frac{36}{35}\left(s^2 + \frac{abc}{s}\right) \quad \left(s = \frac{a+b+c}{2}\right).$$

6.21 On the interval $-1 \leqq x \leqq +1$, let there be given n distinct points P_1, P_2, . . . , P_n. Let T_k be the product of the distances of the point P_k from the other points. Prove that

$$\sum_{k=1}^{n} \frac{1}{T_k} \geqq 2^{n-2}.$$

6.22 1°: If a, b, c are the lengths of the sides of any triangle (which is nondegenerate), prove that

$$\tfrac{1}{2}a^2 < b^2+c^2 \leqq 2a^2, \tag{1}$$

where $a = \max(a, b, c)$.

2°: Are the conditions (1) sufficient for the existence of a triangle (which is nondegenerate) with sides of lengths a, b, c?

SOLUTION. 1°:

$$\{b \leqq a,\ c \leqq a\} \Rightarrow b^2+c^2 \leqq 2a^2, \tag{2}$$

$$a < b+c \Rightarrow a^2 < b^2+2bc+c^2, \tag{3}$$

$$(b-c)^2 \geqq 0 \Rightarrow 2bc \leqq b^2+c^2. \tag{4}$$

From (3) and (4) it follows that

$$a^2 < 2(b^2+c^2) \Rightarrow \tfrac{1}{2}a^2 < b^2+c^2. \tag{5}$$

From (2) and (5), we obtain (1).

2°: If $b = \frac{2}{3}a$, $c = \frac{1}{3}a$, the conditions (1) hold, but $b+c = a$, which contradicts the supposition that the triangle is nondegenerate.

Now, if $b = \frac{7}{9}a$, $c = \frac{1}{9}a$, the conditions (1) are again satisfied. However, with the lengths a, b, c so chosen, it is not possible to construct a triangle, because $b+c < a$. Thus, the conditions (1) do not suffice.

6.23 Prove that if, with $|BC| = a$, $|CA| = b$, $|AB| = c$, it is possible to construct a triangle ABC, then with

$$|\beta\gamma| = \sqrt[n]{a},\ |\gamma\alpha| = \sqrt[n]{b},\ |\alpha\beta| = \sqrt[n]{c}$$

(n a natural number > 1), one may construct a triangle $\alpha\beta\gamma$.

PROOF. Without loss of generality, we may suppose that

$$0 < a \leqq b \leqq c. \tag{1}$$

Since a, b, c are the lengths of the sides of the triangle,

$$c < a+b, \tag{2}$$

$$c > b-a. \tag{3}$$

The relation (1) for the triangle $\alpha\beta\gamma$ corresponds to

$$0 < \sqrt[n]{a} \leqq \sqrt[n]{b} \leqq \sqrt[n]{c}. \tag{4}$$

The problem may be solved if we can prove that the inequalities (2) and (3) imply that

$$\sqrt[n]{c} < \sqrt[n]{a}+\sqrt[n]{b}, \text{ and,} \tag{5}$$

$$\sqrt[n]{c} > \sqrt[n]{b}-\sqrt[n]{a}. \tag{6}$$

Let us suppose that (5) is false, so that

$$\sqrt[n]{c} \geqq \sqrt[n]{a}+\sqrt[n]{b}. \tag{7}$$

Then, after raising both sides to the n^{th} power, we find

$$c \geqq a+b+M, \tag{8}$$

where M is a positive number.

Since (8) contradicts (2) for all a and b, we have proved that (5) is valid.

Let us assume now that (6) does not hold, but instead that

$$\sqrt[n]{c} \leqq \sqrt[n]{b}-\sqrt[n]{a}.$$

Then, after raising to the n^{th} power, it follows that

$$a+c+M \leqq b, \tag{9}$$

where M is a positive number.

Since (9) contradicts (3), we have proved the inequality (6). Consequently, if the inequalities (2) and (3) are valid, (5) and (6) follow.

REMARK. For the six relations

$$a+b > c, \quad b+c > a, \quad c+a > b,$$

$$a > |b-c|, \quad b > |c-a|, \quad c > |a-b|,$$

which hold for the sides a, b, c of a triangle, only three are independent, the others following from these. If we use the assumption (1), only $a+b>c$ needs to be checked. This means that in the proof it is possible to omit the relations (3) and (6).

§ 7. Inequalities Involving Mean Values and Symmetric Functions

7.1 Given positive numbers a and b ($> a$), let

$$A = \tfrac{1}{2}(a+b),\ G = \sqrt{ab},\ H = \frac{2ab}{a+b}.$$

Prove the following inequalities:

$$1°:\ A > G > H,$$
$$2°:\ A-G > G-H,$$
$$3°:\ A-G < (b-a)^2/(8a),$$
$$4°:\ A-H < (b-a)^2/(4a).$$

PROOF. 1°: Since

$$A-G = \tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2 > 0, \tag{1}$$

we have $A > G$. Since

$$H-G = -\frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 < 0, \tag{2}$$

we have $H < G$.

2°: From (1) and (2) it follows that

$$A-2G+H = \frac{(\sqrt{a}-\sqrt{b})^4}{2(a+b)} > 0 \Rightarrow A-G > G-H.$$

3°: From (1) we obtain

$$A-G = \frac{(a-b)^2}{2(\sqrt{a}+\sqrt{b})^2},$$

whence, by the assumption that $a < b$, it follows that

$$A-G < (a-b)^2/(8a).$$

4°: In a similar manner, we have

$$A-H = \frac{(a-b)^2}{2(a+b)} \Rightarrow A-H < (a-b)^2/(4a).$$

7.2 Prove the inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{a^8+b^8+c^8}{a^3b^3c^3} \quad (a, b, c \text{ distinct and positive}).$$

7.3 Let $a_1, a_2, \ldots, a_n$ be positive numbers no two of which are equal. Prove the inequalities

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}>\frac{n}{s}, \quad \frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{a_k}>\sum_{k=1}^{n}\frac{1}{s-a_k} \quad \left(s=\sum_{k=1}^{n}a_k\right).$$

7.4 If $a_1, a_2, \ldots, a_n > 0$ and $p_k = \left(\dfrac{a_1^k+a_2^k+\ldots+a_n^k}{n}\right)^{1/k}$,

then

$$p_r \geqq p_s \quad (r > s).$$

7.5 Let the sum of four positive numbers be $4c$ and the sum of their squares $8c^2$; show that none of these numbers can exceed $(1+\sqrt{3})c$.

SOLUTION. Since the arithmetic mean of the squares is not less than the square of the arithmetic mean,

$$\left(\frac{4c-a_1}{3}\right)^2=\left(\frac{a_2+a_3+a_4}{3}\right)^2\leqq\frac{a_2^2+a_3^2+a_4^2}{3}=\frac{8c^2-a_1^2}{3},$$

i.e.,

$$a_1^2-2ca_1-2c^2 \leqq 0 \Rightarrow a_1 \leqq c(1+\sqrt{3}).$$

(Solution by D. Djoković.)

7.6 Use Bernoulli's inequality,

$$(1+a)^n \geqq 1+na \quad (a > -1, n = 1, 2, 3, \ldots), \tag{1}$$

to prove

$$b^{1/m} \leqq 1+\frac{b-1}{m} \quad (b > 0; m = 1, 2, 3, \ldots), \text{ and} \tag{2}$$

$$(a^{m-1}b)^{1/m} \leqq \frac{(m-1)a+b}{m} \quad (a > 0, b > 0). \tag{3}$$

Proof. From (1) we have

$$\left(1+\frac{b-1}{m}\right)^m \geqq 1+m\frac{b-1}{m} = b, \tag{4}$$

if $(b-1)/m > -1$ and $m = 1, 2, 3, \ldots$, i.e., if $b > 1-m$ and $m = 1, 2, 3, \ldots$. The inequality (2) follows at once.

To prove (3), we replace b in (2) by b/a to obtain

$$(b/a)^{1/m} \leqq 1+\frac{(b/a)-1}{m}. \tag{5}$$

Multiplying both sides of (5) by a yields

$$(a^{m-1}b)^{1/m} \leqq a+\frac{b-a}{m} = \frac{(m-1)a+b}{m}$$

which is (3).

7.7 Prove the inequality

$$3(a^3+b^3+c^3) \geqq (a^2+b^2+c^2)(a+b+c) \quad (a, b, c > 0)$$

and from this result deduce that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geqq \frac{3}{2} \quad (a, b, c > 0).$$

7.8 Let $b_1, b_2, \ldots, b_n$ be any permutation of the positive numbers $a_1, a_2, \ldots, a_n$; then show that

$$\sum_{k=1}^{n} a_k/b_k \geqq n.$$

Solution. This result is an immediate consequence of the inequality between the arithmetic and geometric means, since

$$\prod_{k=1}^{n} \frac{a_k}{b_k} = 1.$$

7.9 Prove the inequality

$$pa+qb+rc \geqq a^p b^q c^r \; (a, b, c, p, q, r > 0;\; p+q+r = 1). \tag{1}$$

PROOF. Applying the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{n}\sum_{\nu=1}^{n} f(x_\nu) \geqq \left\{\prod_{\nu=1}^{n} f(x_\nu)\right\}^{1/n} = \exp\left(\frac{1}{n}\sum_{\nu=1}^{n}\log f(x_\nu)\right),$$

whence, taking limits as $n \to \infty$,

$$\int_\alpha^\beta f(x)\mathrm{d}x \geqq \exp\int_\alpha^\beta \log f(x)\mathrm{d}x, \tag{2}$$

where we assume that $f(x)$ is positive and piecewise continuous over the interval $[\alpha, \beta]$.

In particular, if we set

$$f(x) = \begin{cases} a & (0 < x < p), \\ b & (p < x < p+q), \\ c & (p+q < x < 1), \end{cases}$$

in (2) we obtain the inequality (1).

(Proof by D. Djoković.)

7.10 If $p_0 = 1$, $p_k = \dfrac{\sigma_k}{\binom{n}{k}}$ $(k = 1, 2, \ldots, n)$, where σ_k is the elementary symmetric function of order k of the positive numbers $a_1, a_2, \ldots, a_n$, then show that

$$p_{r-1}p_{r+1} \leqq p_r^2 \quad (1 \leqq r \leqq n-1),$$
$$p_1 \geqq p_2^{1/2} \geqq \cdots \geqq p_r^{1/r} \geqq \cdots \geqq p_n^{1/n}.$$

REMARK. For this result see: J. W. Archbold; *Algebra*, London 1958, p. 53—55.

7.11 Prove the inequality

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leqq \tfrac{1}{2}(a+b+c) \quad (a, b, c > 0). \tag{1}$$

SOLUTION. From the inequalities

$$\frac{b+c}{2} \geqq \sqrt{bc}, \frac{c+a}{2} \geqq \sqrt{ca}, \frac{a+b}{2} \geqq \sqrt{ab}, \tag{2}$$

we obtain

$$\frac{1}{b+c} \leqq \tfrac{1}{2}\frac{1}{\sqrt{bc}}, \frac{1}{c+a} \leqq \tfrac{1}{2}\frac{1}{\sqrt{ca}}, \frac{1}{a+b} \leqq \tfrac{1}{2}\frac{1}{\sqrt{ab}}.$$

It follows that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \leqq \tfrac{1}{2}(\sqrt{bc}+\sqrt{ca}+\sqrt{ab}).$$

Using (2) again, this inequality yields

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \leqq \tfrac{1}{2}\left(\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}\right) = \tfrac{1}{2}(a+b+c).$$

Equality holds in (1) only when $a = b = c$.

7.12 Prove the inequality

$$\tfrac{1}{2}(a^m+b^m) < \{\tfrac{1}{2}(a+b)\}^m \quad (0 < m < 1;\ a, b > 0;\ a \neq b).$$

Generalize.

7.13 Prove that if

$$s_r = \sum_{k=1}^{n} a_k{}^r \text{ and } p_r = \sum_{a_1, a_2, \cdots, a_n} a_1 a_2 \ldots a_r$$

(i.e., p_r is the r-th elementary symmetric function of $a_1, a_2, \ldots, a_n$), then

$$\frac{p_r}{s_r} \leqq \frac{(n-1)!}{r!(n-r)!} \quad (a_1, a_2, \ldots, a_n > 0).$$

7.14 If a, b, c are three distinct real numbers, prove that

$$3\min(a, b, c) < \sum a - (\sum a^2 - \sum ab)^{1/2}$$
$$< \sum a + (\sum a^2 - \sum ab)^{1/2} < 3\max(a, b, c),$$

where

$$\sum a = a+b+c, \sum a^2 = a^2+b^2+c^2, \sum ab = ab+bc+ca.$$

PROOF. The function

$$f(x) = (x-a)(x-b)(x-c) = x^3-(\sum a)x^2+(\sum ab)x-abc$$

vanishes for $x = a$, $x = b$, $x = c$. We deduce that $f'(x)$ vanishes for values of x which are solutions of

$$3x^2-2(\sum a)x+\sum ab = 0.$$

These solutions lie between min (a, b, c) and max(a, b, c). From this we obtain the proposed inequalities.

7.15 If A_k and G_k are the arithmetic and geometric means of the first k numbers in the sequence $a_1, a_2, \ldots$ $(a_i > 0)$, prove that

$$k(A_k-G_k) \geqq (k-1)(A_{k-1}-G_{k-1}).$$

(See the paper of L. Tchakaloff: Sur quelques inégalités entre la moyenne arithmétique et la moyenne géométrique, *Publications de l'Institut mathématique,* Beograd, v. 3(17)1963, p. 43).

7.16 Let σ_k $(k = 1, 2, \ldots, n)$ be the elementary symmetric functions of order k of the real numbers $x_1, x_2, \ldots, x_n$. Prove that

$$\{\sigma_1 > 0, \sigma_2 > 0, \ldots, \sigma_n > 0\} \Leftrightarrow \{x_1 > 0, x_2 > 0, \ldots, x_n > 0\}.$$

EXAMPLE. If x, y, z are real, then

$$\{x+y+z > 0, yz+zx+xy > 0, xyz > 0\} \Leftrightarrow \{x > 0, y > 0, z>0\}.$$

PROOF. We see at once that

$$x_1 > 0, x_2 > 0, \ldots, x_n > 0 \text{ implies } \sigma_1 > 0, \sigma_2 > 0, \ldots, \sigma_n > 0.$$

In order to prove that the converse holds, we form the polynomial equation with roots $x_1, x_2, \ldots, x_n$, namely,

$$x^n-\sigma_1 x^{n-1}+\sigma_2 x^{n-2}- \ldots +(-1)^n\sigma_n = 0.$$

We determine whether this equation may have negative roots by replacing x by $-x$, whence

$$x^n+\sigma_1 x^{n-1}+\sigma_2 x^{n-2}+ \dots +\sigma_n = 0.$$

If the conditions $\sigma_1 > 0, \sigma_2 > 0, \dots, \sigma_n > 0$ hold, the polynomial

$$x^n+\sigma_1 x^{n-1}+\sigma_2 x^{n-2}+ \dots +\sigma_n$$

does not vanish for any positive value of x. Thus, since this polynomial vanishes for $x = -x_k (k = 1, 2, \dots, n)$, it follows that

$$x_k > 0 \quad (k = 1, 2, \dots, n).$$

Thus

$\sigma_1 > 0, \sigma_2 > 0, \dots, \sigma_n > 0$ implies $x_1 > 0, x_2 > 0, \dots, x_n > 0$.

7.17 1°: Does

(A) $$x > 0, y > 0, z > 0$$

imply

(B) $$x+y+z > 0, \ xyz > 0?$$

2°: Does (B) imply (A)?
3°: Are (A) and (B) equivalent?

7.18 Prove the inequality

$$\sum_{\substack{i,k=1\\ i\leq k}}^{n} x_i x_k \geq 0 \quad (x_1, x_2, \dots, x_n \text{ real}).$$

Proof. This inequality follows directly from the identity

$$2\sum_{\substack{i,k=1\\ i\leq k}}^{n} x_i x_k = \left(\sum_{k=1}^{n} x_k\right)^2 + \sum_{k=1}^{n} x_k^2.$$

7.19 Prove the inequality $(a+b+c)(bc+ca+ab) > 9\,abc$, where a, b, c are positive numbers which are not all equal.

Proof. $(a+b+c)(bc+ca+ab)-9\,abc$

$$= a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-6\,abc$$
$$= a(b-c)^2+b(c-a)^2+c(a-b)^2 > 0.$$

REMARK. What changes must be made in the given inequality if a, b, c are negative numbers?

7.20 $\left(\sum_{k=0}^{n-1} (n-k)d_k\right)^n \geqq n^n \prod_{k=0}^{n-1} \sum_{\nu=0}^{k} d_\nu \quad (d_k \geqq 0).$

7.21 $\sum_{k=1}^{n} x_k \geqq n \quad \left(x_k > 0,\ \prod_{k=1}^{n} x_k = 1\right).$

7.22 $(b+c)(c+a)(a+b) \geqq 8\,abc \quad (a, b, c \geqq 0).$

7.23 $bc(b+c)+ca(c+a)+ab(a+b) \geqq 6\,abc \quad (a, b, c \geqq 0).$

7.24 $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geqq 8 \quad (a, b, c > 0;\ a+b+c=1).$

7.25 $b^2c^2+c^2a^2+a^2b^2 \geqq abc(a+b+c).$

7.26 $(a^2+b^2+c^2+d^2+e^2)(a^3+b^3+c^3+d^3+e^3) \geqq 25\,abcde$
$(a, b, c, d, e \geqq 0).$

7.27 $a^3+b^3+c^3+15abc \leqq 2(a+b+c)(a^2+b^2+c^2) \quad (a, b, c \geqq 0).$

7.28 $(s-a)(s-b)(s-c)(s-d) \geqq 81abcd \ (a, b, c, d > 0;$
$s = a+b+c+d).$

7.29 $\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geqq 2 \quad (a, b, c, d > 0).$

7.30 $\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd} \leqq \frac{3}{2}(a+b+c+d)$
$(a, b, c, d \geqq 0).$

7.31 $\sum_{k=1}^{n} \frac{s}{s-a_k} > \frac{n^2}{n-1} \quad \left(a_k > 0;\ s = \sum_{k=1}^{n} a_k\right).$

7.32 $\sum_{k=1}^{n} \frac{a_k}{s-a_k} > \frac{n}{n-1} \quad \left(a_k > 0;\ s = \sum_{k=1}^{n} a_k\right).$

7.33 $\sum_{k=1}^{n-1} (a_k a_{k+1})^{1/2} \leqq \frac{1}{2}(n-1) \sum_{k=1}^{n} a_k \quad (a_k \geqq 0).$

7.34 $a+b+c \geqq a^p b^q c^r + a^r b^p c^q + a^q b^r c^p$
$(a, b, c, p, q, r > 0;\ p+q+r = 1)$.

7.35 $(2n+1)x^n \leqq 1+x+x^2+\ldots+x^{2n} \quad (x \geqq 0)$.

7.36 $1-x^{2n} \geqq 2nx^n(1-x) \quad (0 \leqq x \leqq 1)$.

7.37 $(abc)^5(a^3+b^3+c^3)^5 \leqq (b^5c^5+c^5a^5+a^5b^5)(a^5+b^5+c^5)^4$
$(a, b, c \geqq 0)$.

7.38 $\left(\frac{1}{n}\sum_{k=1}^{n} a_k{}^p\right)^{1/p} \leqq \left(\frac{1}{n}\sum_{k=1}^{n} a_k{}^q\right)^{1/q} \quad (0 < p < q;\ a_k \geqq 0)$.

7.39 $\prod_{i=1}^{k}\left(\frac{1}{n}\sum_{j=1}^{n} a_j{}^{\alpha_i}\right) \leqq \frac{1}{n}\sum_{j=1}^{n} a_j{}^m \left(m = \sum_{i=1}^{k} \alpha_i;\ \alpha_i, a_j > 0\right)$.

7.40 $2(n-1)(\sum x_1{}^2)(2\sum x_1{}^2x_2{}^2 + \sum x_1{}^2x_2x_3) - n(\sum x_1{}^2x_2)^2 \geqq 0$
($x_1, x_2, \ldots, x_n$ real numbers).

7.41 $\frac{3}{2} \leqq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leqq 2$
(where a, b, c are the sides of a triangle).

7.42 $\frac{n}{n-1} \leqq \sum_{i=1}^{n} \frac{x_i}{\sum_{\substack{\nu=1 \\ \nu\neq i}}^{n} x_\nu} \leqq \frac{n-1}{n-2} \quad \left(x_i \leqq \frac{1}{n-1}\sum_{i=1}^{n} x_i\right)$.

7.43 $\sum_{k=1}^{n} a_k{}^m \geqq n \quad \left(m > 1,\ a_k > 0,\ \sum_{k=1}^{n} a_k = n\right)$.

7.44 $(n-1)\left(\sum_{k=1}^{n} a_k\right)^2 \geqq 2n \sum_{\substack{r,s=1 \\ r<s}}^{n} a_r a_s$.

§ 8. The Inequalities of Cauchy-Schwarz-Buniakowski, Hölder, Minkowski and Chebychev.

8.1 $(a\alpha+b\beta)^2 \leqq (a^2+b^2)(\alpha^2+\beta^2).$

8.2 $(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \geqq 9a^2b^2c^2 \quad (a, b, c \geqq 0).$

8.3 $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geqq n^2 \quad (a_k > 0).$

8.4 $\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leqq \sum_{k=1}^{n} k a_k^2 \sum_{k=1}^{n} \frac{b_k^2}{k}.$

8.5 $\left(\sum_{k=1}^{n} a_k^m\right)^2 \leqq \left(\sum_{k=1}^{n} a_k^{m+s}\right)\left(\sum_{k=1}^{n} a_k^{m-s}\right) \quad (a_k > 0).$

8.6 $\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leqq \sum_{k=1}^{n} a_k^4 \sum_{k=1}^{n} b_k^4 \left(\sum_{k=1}^{n} c_k^2\right)^2.$

8.7 $\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 \leqq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2.$

8.8 $\left(\sum_{k=1}^{n} \frac{a_k}{k}\right)^2 \leqq \sum_{k=1}^{n} k^3 a_k^2 \sum_{k=1}^{n} \frac{1}{k^5}.$

8.9 $\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leqq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \quad (a_k, b_k \text{ complex numbers}).$

8.10 $\left|\sum_{k=1}^{n} a_k b_k\right| \leqq 1 \quad \left(\sum_{k=1}^{n} |a_k|^2 = \sum_{k=1}^{n} |b_k|^2 = 1\right)$

$(a_k, b_k \text{ complex numbers}).$

8.11 $(a_1+a_2+a_3)(c_1+c_2+c_3)-(b_1+b_2+b_3)^2 > 0$

$(a_1, a_2, a_3, a_1c_1-b_1^2, a_2c_2-b_2^2, a_3c_3-b_3^2 > 0).$

8.12 $\sum_{k=1}^{n} \frac{1}{a_k} \geqq n^2 \left(\sum_{k=1}^{n} a_k \leqq 1;\ a_k > 0\right).$

8.13 $(a_1+b_1)^p(a_2+b_2)^q \geqq a_1^p a_2^q + b_1^p b_2^q$

$(p+q = 1;\ p, q, a_k, b_k \geqq 0).$

8.14 $(1+x)^r(1+y)^{1-r} \geqq 1+x^r y^{1-r} \quad (x, y \geqq 0;\ 0 < r < 1).$

8.15 $[(a_1+b_1)^r+(a_2+b_2)^r]^{1/r} \leqq (a_1{}^r+a_2{}^r)^{1/r}+(b_1{}^r+b_2{}^r)^{1/r}$
$(r > 1;\ a_k,\ b_k \geqq 0)$.

8.16 $\prod_{k=1}^{n}(1+a_k) \geqq (1+b)^n \quad \left(a_k > 0;\ \prod_{k=1}^{n} a_k = b^n\right)$.

8.17 $\prod_{k=1}^{n}(1+a_k) \geqq 2^n \quad \left(a_k > 0;\ \prod_{k=1}^{n} a_k = 1\right)$.

8.18 $[(a+b)(\alpha+\beta)]^{1/2} \geqq (a\alpha)^{1/2}+(b\beta)^{1/2} \quad (a,\ b,\ \alpha,\ \beta \geqq 0)$.

8.19 $[(a_1+b_1)(a_2+b_2)(a_3+b_3)]^{1/3} \geqq (a_1a_2a_3)^{1/3}+(b_1b_2b_3)^{1/3}$
$(a_k,\ b_k \geqq 0;\ k = 1,\ 2,\ 3)$.

8.20 $[(a_1+b_1)(a_2+b_2)(a_3+b_3)(a_4+b_4)]^{1/4}$
$\geqq (a_1a_2a_3a_4)^{1/4}+(b_1b_2b_3b_4)^{1/4} \quad (a_k,\ b_k \geqq 0;\ k = 1,2,3,4)$.

8.21 $\left|\sum_{k=m}^{n}\frac{a_k}{k^{1/3}}\right|^3 \leqq \left(\sum_{k=m}^{n}|a_k|^{3/2}\right)^2 \sum_{k=m}^{n}\frac{1}{k} \quad (m \leqq n)$.

8.22 $\left(\sum_{k=1}^{n} a_kb_kc_k\right)^6 \leqq \sum_{k=1}^{n} a_k{}^4 \sum_{k=1}^{n} a_k{}^2 \sum_{k=1}^{n} b_k{}^4 \sum_{k=1}^{n} b_k{}^2 \sum_{k=1}^{n} c_k{}^4 \sum_{k=1}^{n} c_k{}^2$.

8.23 $\frac{1}{n}\sum_{k=1}^{n} a_k{}^{p+q} \geqq \left(\frac{1}{n}\sum_{k=1}^{n} a_k{}^p\right)\left(\frac{1}{n}\sum_{k=1}^{n} a_k{}^q\right) \quad (p,\ q > 0;\ a_k \geqq 0)$.

8.24 $\left(\frac{1}{n}\sum_{\nu=1}^{n} a_\nu\right)^k \leqq \frac{1}{n}\sum_{\nu=1}^{n} a_\nu{}^k \quad (a_\nu \geqq 0)$.

8.25 $\left(\sum_{k=1}^{n} a_k\right)^2 \leqq n\sum_{k=1}^{n} a_k{}^2$.

8.26 $bc(b+c)+ca(c+a)+ab(a+b) \leqq 2(a^3+b^3+c^3) \quad (a,\ b,\ c \geqq 0)$.

8.27 $(bc+ca+ab)(a+b+c)^4 \leqq 27(a^3+b^3+c^3)^2 \quad (a,\ b,\ c \geqq 0)$.

8.28 $(a^4+b^4+c^4+d^4)(a^3+b^3+c^3+d^3) \leqq 4(a^7+b^7+c^7+d^7)$
$(a,\ b,\ c,\ d \geqq 0)$.

8.29 $(a+b)(a^2+b^2)(a^3+b^3) \leqq 4(a^6+b^6) \quad (a,\ b \geqq 0)$.

8.30 $(a+b)(a^3+b^3)(a^7+b^7) \leqq 4(a^{11}+b^{11}) \quad (a,\ b \geqq 0)$.

8.31 $(a^2+b^2)(a^3+b^3)(a^6+b^6) \leqq 4(a^{11}+b^{11}) \quad (a, b \geqq 0)$.

8.32 $ab(a^2+b^2) \leqq a^4+b^4$.

8.33 $a^2b^2(a^5+b^5) \leqq a^9+b^9 \quad (a, b \geqq 0)$.

8.34 $n(1+x^{n-1}) \geqq 2\dfrac{x^n-1}{x-1} \quad (x > 0;\ n \geqq 2)$.

8.35 $(a+b)^n < 2^{n-1}(a^n+b^n) \quad (n \geqq 2;\ a, b > 0;\ a \neq b)$.

8.36 $a^n-1 \geqq n(a^{(n+1)/2}-a^{(n-1)/2}) \quad (a \geqq 1)$.

8.37 $\dfrac{a}{\sqrt{b}}+\dfrac{b}{\sqrt{a}} \geqq \sqrt{a}+\sqrt{b} \quad (a, b > 0)$.

8.38 $(a^4+b^4)(a^5+b^5) < 2(a^9+b^9) \quad (a, b > 0;\ a \neq b)$.

8.39 $(a^3+b^3)(a^2+b^2) < 2(a^5+b^5) \quad (a, b > 0;\ a \neq b)$.

§ 9. Inequalities Involving Integrals.

9.1 Let

$$I_n = 2n \cot^n a \int_0^a \tan^n x \, dx \quad \left(0 < a < \frac{\pi}{2};\ n \text{ a natural number}\right).$$

Prove the inequality

$$\int_0^a \tan^n x \, dx \leqq \int_0^a \tan^n x \sec^2 x \, dx, \tag{1}$$

and hence deduce that

$$I_n < 2 \tan a. \tag{2}$$

Will the inequalities (1) and (2) be valid if n is an arbitrary real number?

9.2 Prove the inequality

$$q(x^p-1) \geqq p(x^q-1) \quad (p > q > 0;\ x > 0)$$

and from this result, by integration, prove the inequality

$$\frac{1}{p}\left\{\frac{x^p}{(p+1)^n}-1\right\} \geqq \frac{1}{q}\left\{\frac{x^q}{(q+1)^n}-1\right\} \quad (n \text{ a natural number}).$$

9.3 Let

$$I_n = \int_0^a \tan^n x \, dx \quad (n = 2, 3 \ldots; \quad 0 < a < \tfrac{1}{2}\pi).$$

Prove the relation

$$I_n + I_{n-2} = \frac{1}{n-1} \tan^{n-1} a.$$

Compute I_3 and I_4, and thereby deduce that

$$a + \log \sec a < \tan a + \tfrac{1}{2} \tan^2 a - \tfrac{1}{3} \tan^3 a \quad (0 < a < \pi/4).$$

9.4 Prove Esseen's inequality

$$\phi(-a-b) \leqq 2\phi(-a)\, \phi(-b) \quad (a, b \geqq 0), \tag{1}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} \, dt. \tag{2}$$

PROOF. Consider the function

$$F(a, b) = \phi(-a-b)/\phi(-a) \quad (a, b \geqq 0). \tag{3}$$

Differentiating we obtain

$$\begin{aligned} \frac{\partial F}{\partial a} &= \frac{1}{\phi^2(-a)} \{\phi'(-a)\phi(-a-b) - \phi'(-a-b)\phi(-a)\} \\ &= \frac{e^{-a^2/2}}{\sqrt{2\pi}\phi^2(-a)} \{\phi(-a-b) - e^{-\frac{1}{2}b^2 - ab}\, \phi(-a)\} \\ &= \frac{e^{-a^2/2}}{\sqrt{2\pi}\phi^2(-a)} G(a, b), \end{aligned} \tag{4}$$

where we have set

$$G(a, b) = \phi(-a-b) - e^{-\frac{1}{2}b^2 - ab}\, \phi(-a) \quad (a, b \geqq 0). \tag{5}$$

From this result we find

$$\frac{\partial G}{\partial a} = b e^{-\frac{1}{2}b^2 - ab}\, \phi(-a) \geqq 0,$$

whence

$$G(a, b) \leqq \lim_{a \to +\infty} G(a, b) = 0,$$

and thus (4) gives

$$\frac{\partial F}{\partial a} \leqq 0.$$

Accordingly,

$$F(a, b) \leqq F(0, b) = \frac{\phi(-b)}{\phi(0)} = 2\phi(-b),$$

which completes the proof of Esseen's inequality.
(Proof by D. Djoković.)

REMARK. Starting with Esseen's inequality, prove that for the function

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t,$$

we have the analogous inequality

$$f(a)f(b) \geqq f(a)+f(b)-f(a+b) \quad (a, b \geqq 0).$$

9.5 Let $f(x)$ be a continuous, non-negative function for all real x, and suppose that

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = 1.$$

If

$$g(t) = \int_{-\infty}^{+\infty} f(x) \cos xt \, \mathrm{d}x \quad (t \text{ real}),$$

prove the inequality

$$g(2t) > 2\{g(t)\}^2 - 1 \quad (t \neq 0).$$

PROOF. Since

$$\cos 2xt = 2 \cos^2 xt - 1,$$

$$g(2t) = 2 \int_{-\infty}^{+\infty} f(x) \cos^2 xt \, \mathrm{d}x - 1,$$

whence the given inequality may be written in the form

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x \int_{-\infty}^{+\infty} f(x) \cos^2 xt \, \mathrm{d}x > \left\{ \int_{-\infty}^{+\infty} f(x) \cos xt \, \mathrm{d}x \right\}^2.$$

Since the functions $\sqrt{f(x)}$ and $\sqrt{f(x)}\cos xt$ $(t \neq 0)$ are linearly independent, we see that the preceding inequality follows from the Cauchy-Schwarz-Buniakowski inequality

$$\int_a^b f_1^2(x)\mathrm{d}x \int_a^b f_2^2(x)\mathrm{d}x > \left\{\int_a^b f_1(x)\, f_2(x)\,\mathrm{d}x\right\}^2,$$

where $f_1(x)$ and $f_2(x)$ are two continuous, linearly independent functions, when we set

$$a = -\infty,\, b = +\infty,\, f_1(x) = \sqrt{f(x)},\, f_2(x) = \sqrt{f(x)}\cos xt\ \ (t \neq 0).$$

(Solution by D. Djoković.)

9.6 Prove the inequality

$$\frac{\pi}{2n} > \left(\int_0^{\pi/2} \sin^n \theta\,\mathrm{d}\theta\right)^2 > \frac{\pi}{2(n+1)} \quad (n \text{ a natural number}).$$

9.7 Let the function $f(x)$ be positive and non-increasing over the interval $[1, +\infty)$. Show that if

$$g_n(t) = t^n f(t^n) \quad (t > 1;\, n = 0, 1, 2, \ldots),$$

then

$$\frac{t-1}{t}\, g_{n+1}(t) \leqq \int_{t^n}^{t^{n+1}} f(x)\,\mathrm{d}x \leqq (t-1)g_n(t).$$

9.8 Prove the inequality

$$\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x \geqq \sqrt{n}\,\frac{(2n)!!}{(2n+1)!!} \quad (n = 1, 2, \ldots). \tag{1}$$

Proof. Start with the following pair of inequalities:

$$\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x \geqq \sqrt{n}\int_0^1 e^{-nx^2}\,\mathrm{d}x = \int_0^{\sqrt{n}} e^{-x^2}\,\mathrm{d}x \quad (n = 1, 2, \ldots); \tag{2}$$

$$1-x^2 \leqq e^{-x^2}. \tag{3}$$

From (3) there follow in succession

$$(1-x^2)^n \leqq e^{-nx^2} \quad (n = 1, 2, \ldots), \tag{4}$$

$$\int_0^1 (1-x^2)^n\,\mathrm{d}x \leqq \int_0^1 e^{-nx^2}\,\mathrm{d}x. \tag{5}$$

From (2) and (5) we find

$$\int_0^{+\infty} e^{-x^2}\,dx \geqq \sqrt{n}\int_0^1 (1-x^2)^n\,dx = \sqrt{n}\,\frac{(2n)!!}{(2n+1)!!}.$$

9.9 Prove the inequality

$$\int_0^{+\infty} e^{-x^2}\,dx \leqq \sqrt{n}\,\frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2} \quad (n = 1, 2, \ldots). \tag{1}$$

PROOF. Starting with the inequality

$$e^{-x^2} \leqq \frac{1}{1+x^2} \quad (x \geqq 0),$$

we obtain

$$e^{-nx^2} \leqq \frac{1}{(1+x^2)^n} \quad (x \geqq 0;\ n = 1, 2, \ldots),$$

$$\int_0^{+\infty} e^{-nx^2}\,dx \leqq \int_0^{+\infty} \frac{1}{(1+x^2)^n}\,dx = \int_0^{\pi/2} \cos^{2n-2} t\,dt = \frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2}.$$

If we set $x\sqrt{n} = y$, the preceding inequality yields (1) at once.

9.10 Verify that

$$\int_0^1 \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \tfrac{1}{2}\pi(\sqrt{2}-1)$$

and from this result deduce the inequality

$$\tfrac{1}{2}\pi(\sqrt{2}-1) < \int_0^1 \frac{\sqrt{1-x^2}}{1+x^4}\,dx < \tfrac{1}{4}\pi.$$

9.11 If $f_0(x) > 0 (x \geqq 0)$ and $f_n(x) = \int_0^x f_{n-1}(t)dt \quad (n = 1, 2, \ldots)$, prove the inequality

$$\frac{f_n(x)}{x^n} > \frac{f_{n+1}(x)}{x^{n+1}}.$$

9.12 Prove the inequality

$$\int_a^{+\infty} e^{-x^2}\,dx < \frac{1}{a}\,e^{-a^2} \quad (a > 0). \tag{1}$$

PROOF. If $x > a > 0$, then

$$e^{-x^2} < e^{-ax}, \text{ since } x^2 > ax.$$

Since

$$\int_a^b e^{-x^2}\, dx < \int_a^b e^{-ax}\, dx \quad (b > a > 0),$$

one has

$$\lim_{b\to+\infty} \int_a^b e^{-x^2}\, dx < \lim_{b\to+\infty} \int_a^b e^{-ax}\, dx \quad (a > 0),$$

whence the inequality (1) follows.

9.13 Prove the inequality

$$(2\pi)^{-1/2} \int_{-a}^{+a} e^{-x^2/2}\, dx < (1-e^{-a^2})^{1/2} \quad (a > 0).$$

9.14 Prove the inequality

$$\int_0^1 \frac{1}{(1-2x\cos t+x^2)^{3/2}}\, dx \leqq \frac{\pi^2}{2t^2} \quad (0 < t \leqq \tfrac{1}{2}\pi).$$

HINT. Start with the equality

$$\int_0^1 \frac{1}{(1-2x\cos t+x^2)^{3/2}}\, dx = \int_0^1 \frac{1}{[\sin^2 t+(x-\cos t)^2]^{3/2}}\, dx$$

$$= \int_{x=0}^{x=1} \frac{1}{[\sin^2 t+(x-\cos t)^2]^{3/2}}\, d(x-\cos t) \quad (t \text{ fixed})$$

$$= \frac{1}{\sin^2 t}(\sin \tfrac{1}{2}t+\cos t).$$

REMARK. It is also true that

$$\int_0^1 \frac{1}{(1-2x\cos t+x^2)^{3/2}}\, dx \leqq \frac{9\pi^2}{32\, t^2} \quad (0 < t \leqq \pi/2).$$

9.15 If

$$I = \int_0^1 \frac{1}{\{(1-x^2)(1-a^2x^2)\}^{1/2}}\, dx \quad (a^2 < 1),$$

prove the inequality

$$\sqrt{2} < I < \frac{2}{(1-a^2)^{1/2}}$$

as well as the sharper inequality

$$\frac{\pi}{2} < I < \frac{\pi}{2}\,\frac{1}{(1-a^2)^{1/2}}.$$

9.16 Let $f(t)$ be a non-negative function defined on the interval $[0, 1]$, and let

$$\frac{f(t_1)+f(t_2)}{2} \leqq f\left(\frac{t_1+t_2}{2}\right) \qquad (t_1, t_2 \in [0, 1]).$$

Prove the inequalities:

$$\int_0^1 f(t)\,\mathrm{d}t \leqq 3^n \int_0^1 t^n f(t)\mathrm{d}t \qquad (n = 0, 1, 2, \ldots),$$

$$\frac{2}{(n+1)(n+2)}\int_0^1 f(t)\mathrm{d}t \leqq \int_0^1 t^n f(t)\mathrm{d}t \leqq \frac{2}{n+2}\int_0^1 f(t)\mathrm{d}t.$$

These inequalities were stated and proved by B. Sendov and D. Skordev in the journal: *Fiziko-matematičesko spisanie,* Sofija, v.2, 1959, p.240 and v. 3, 1960, p. 55.

9.17 Prove the inequality

$$\int_a^b |f(x)-g(x)|^2\,\mathrm{d}x \leqq 2\int_a^b |f(x)-h(x)|^2\,\mathrm{d}x + 2\int_a^b |g(x)-h(x)|^2\,\mathrm{d}x. \tag{1}$$

Hint. $f(x)-g(x) = \{f(x)-h(x)\}-\{g(x)-h(x)\}$,

$$|f(x)-g(x)| \leqq |f(x)-h(x)|+|g(x)-h(x)|,$$

$$|f(x)-g(x)|^2 \leqq |f(x)-h(x)|^2+|g(x)-h(x)|^2 + 2|f(x)-h(x)|\,|g(x)-h(x)|.$$

Since $2|f(x)-h(x)|\,|g(x)-h(x)| \leqq |f(x)-h(x)|^2+|g(x)-h(x)|^2$, the preceding inequality becomes

$$|f(x)-g(x)|^2 \leqq 2|f(x)-h(x)|^2+2|g(x)-h(x)|^2.$$

Remark. What conditions must $f(x)$, $g(x)$, $h(x)$ satisfy, in order that the inequality (1) is meaningful?

9.18 Prove the inequality

$$1.462 < \int_0^1 e^{x^2}\,dx < 1.463.$$

9.19 Starting with Jordan's inequality

$$\frac{\sin x}{x} > \frac{2}{\pi} \quad (0 < x < \pi/2), \tag{1}$$

prove the inequality

$$\int_0^{\pi/2} e^{-R\sin x}\,dx < \frac{\pi}{2R}(1-e^{-R}) \quad (R > 0).$$

PROOF. For $0 < x < \pi/2$, (1) implies each of the following:

$$\sin x > (2/\pi)x, \quad -R\sin x < -(2/\pi)xR, \quad e^{-R\sin x} < e^{-2xR/\pi},$$

$$\int_0^{\pi/2} e^{-R\sin x}\,dx < \int_0^{\pi/2} e^{-2xR/\pi}\,dx = e^{-2xR/\pi}\,(-\pi/2R)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2R}(1-e^{-R}).$$

§ 10. Inequalities in the Complex Domain.

10.1 Prove the inequality

$$|a+b| + |a-b| \geqq |a| + |b|,$$

where a and b are complex numbers.

PROOF. $2|a| = |(a+b)+(a-b)| \leqq |a+b|+|a-b|$,
$2|b| = |(a+b)-(a-b)| \leqq |a+b|+|a-b|$;

adding and dividing by 2 gives the stated inequality.

10.2 Show that the inequalities

$$\operatorname{Re} z < 1/2 \text{ and } \left|\frac{z}{1-z}\right| < 1$$

are equivalent.

10.3 Show that the inequalities

$$\operatorname{Re}\frac{1}{1-z} < 1/2 \text{ and } |z| > 1$$

are equivalent.

10.4 Show that the relations

$$\left|\frac{z-a}{1-\bar{a}z}\right| < 1, \left|\frac{z-a}{1-\bar{a}z}\right| = 1, \left|\frac{z-a}{1-\bar{a}z}\right| > 1 \quad (|a| < 1)$$

are equivalent, respectively, to

$$|z| < 1, |z| = 1, |z| > 1.$$

10.5 Prove the inequality

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leqq \left(\sum_{k=1}^{n} a_k^2\right) \cdot \frac{1}{2} \left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right),$$

where the a_k are real and the z_k complex numbers.

REMARK. This inequality is stronger than Schwarz's inequality

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leqq \left(\sum_{k=1}^{n} a_k^2\right) \left(\sum_{k=1}^{n} |z_k|^2\right),$$

since

$$\left|\sum_{k=1}^{n} z_k^2\right| \leqq \sum_{k=1}^{n} |z_k|^2.$$

10.6 Prove the inequality

$$|\log(1+z)| \geqq \log(1+|z|) \quad (z = x+iy),$$

where $\log z$ is the principal value of the logarithm.

10.7 Prove Kloosterman's inequality

$$|e^z - (1+z/n)^n| < \frac{|z|^2}{2n} e^{|z|} \quad (z = x+iy \neq 0; n \text{ a natural number}).$$

REMARK. See: *Wiskundige opgaven met de oplossingen*, Achttiende Deel, eerste stuk, 1943, p. 70–72.

10.8 Prove or disprove the inequality

$$|z^{n+1}-1| > |z|^n\,|z-1| \quad (z = x+iy;\ \mathrm{Re}\, z \geqq 1). \tag{1}$$

PROOF. Note that we must have $z \neq 1$. Let $r = |z|$, $\theta = \arg z$. By reasons of symmetry, we may take $0 \leqq \theta < \pi/2$. Since

$$|z|^{n+1}-1 > |z|^{n+1}-|z|^n \quad (r > 1),$$

the inequality (1) holds if $2\pi-(n+1)\theta \geqq \theta$, i.e., if

$$\theta \leqq \frac{2\pi}{n+2}. \tag{2}$$

For $n = 1$ and $n = 2$, the condition (2) is satisfied for the entire region $D = \{z|\mathrm{Re}\, z \geqq 1,\ z \neq 1\}$, and for these cases the inequality (1) has been proved.

If $n > 2$, the inequality (1) has been proved for the region $E = \{z|\mathrm{Re}\, z \geqq 1,\ z \neq 1,\ |\theta| \leqq 2\pi/(n+2)\}$. In polar coordinates, (1) reads

$$2\, r^{2n+1} \cos \theta + 1 > r^{2n} + 2\, r^{n+1} \cos(n+1)\theta \quad (r \cos \theta \geqq 1). \tag{1'}$$

This relation certainly holds if

$$2\, r^{2n}+1 > r^{2n}+2\, r^{n+1} \Leftrightarrow f(r) = r^{2n}-2\, r^{n+1}+1 > 0. \tag{3}$$

Since the polynomial $f(r)$ has only two positive zeros: $r = 1$ and $r = r_1 > 1$, (3) holds for $r > r_1$.

Since $r_1 > 1$ and $2\, r_1^2 > 1+r_1$, we have

$$r_1^2(r_1^{2n-2}-2\, r_1^{n-1}+1)-(r_1^2-1) = 0,$$
$$r_1^2(r_1-1)(r_1^{n-2}+r_1^{n-3}+ \ldots + 1)^2 = 1+r_1,$$

whence

$$r_1 < 1+\frac{2}{(n-1)^2} = r_0.$$

Consequently, we have proved that the inequality (1) holds for the region $F = \{z|\mathrm{Re}\, z \geqq 1,\ r \geqq r_0\}$. Moreover, we may show that $E \cup F = D$. This relation is equivalent to the inequality

$$\alpha \leqq \frac{2\pi}{n+2} \tag{4}$$

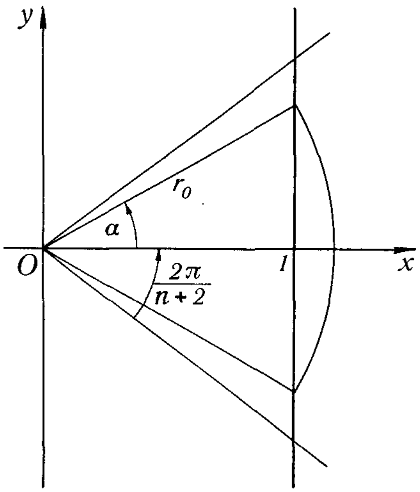

Fig. 8.

where α is the angle shown in Fig. 8. Thus

$$\sin \alpha = \sqrt{1-\frac{1}{r_0^2}} = \frac{2\sqrt{n^2-2n+2}}{n^2-2n+3}, \tag{5}$$

whence

$$\sin \alpha < \frac{2}{\sqrt{n^2-2n+3}} < \frac{2}{n-1}.$$

Applying Jordan's inequality $\sin \theta > 2\theta/\pi$, we obtain

$$\alpha < \frac{\pi}{2}\sin \alpha < \frac{\pi}{n-1},$$

from which we deduce that (4) holds for $n = 4, 5, 6, \ldots$.

For $n = 3$, we find from (5) that $\sin \alpha = \sqrt{5}/3$, so that $\alpha < 49°$ and so (4) is satisfied. This completes the proof. (Solution by D. Djoković.)

10.9 Prove the inequality

$$\frac{|a+b|}{1+|a+b|} \leqq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}. \tag{1}$$

PROOF. Let us assume that (1) does not hold, but rather that

$$\frac{|a+b|}{1+|a+b|} > \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}. \tag{2}$$

Then we find that

$$|a+b| > |a|+|b|+2|ab|+|ab|\,|a+b|, \tag{3}$$

which is impossible because $|a+b| \leqq |a|+|b|$.

REMARK. Is the generalization

$$\frac{\left|\sum_{k=1}^{n} a_k\right|}{1+\left|\sum_{k=1}^{n} a_k\right|} \leqq \sum_{k=1}^{n} \frac{|a_k|}{1+|a_k|}$$

valid?

10.10 Prove that

$$|z-a| \gtreqless |1-\bar{a}z| \Leftrightarrow (1-|a|^2)(|z|^2-1) \gtreqless 0.$$

10.11 If $|a| < 1$ prove the inequality

$$\left|\frac{z-a}{1-\bar{a}z}\right| < \exp\left(\frac{2(1-|a|)}{|1/\bar{z}-a|^2}\right). \tag{1}$$

PROOF. Making use of

$$|z-a|^2 = |1-\bar{a}z|^2+(|z|^2-1)(1-|a|^2)$$

and

$$\log(1+x) \leqq x \quad (x \geqq 0)$$

we find for $|z| \geqq 1$, $|a| < 1$

$$\log\left|\frac{z-a}{1-\bar{a}z}\right| = \tfrac{1}{2}\log\left[1+\frac{(|z|^2-1)(1-|a|^2)}{|1-\bar{a}z|^2}\right]$$

$$\leqq \tfrac{1}{2}\,\frac{(|z|^2-1)(1-|a|^2)}{|1-\bar{a}z|^2}$$

$$< 2\,\frac{1-|a|}{|1/\bar{z}-a|^2}. \tag{2}$$

Hence, (1) holds for $|z| \geqq 1$, $|a| < 1$. On the other hand, we evidently have

$$\frac{|z-a|}{|1-\bar{a}z|} < 1 \quad \text{for } |z| < 1, |a| < 1. \tag{3}$$

By (2) and (3) the given inequality is established.

10.12 Prove that

$$\left| \frac{1}{n} \sum_{\nu=1}^{n} z_\nu \right|^2 \leqq \frac{1}{n} \sum_{\nu=1}^{n} |z_\nu|^2$$

where $z_1, z_2, \ldots, z_n$ are arbitrary complex numbers.

HINT. Use the identity

$$\frac{1}{n} \sum_{\nu=1}^{n} |z_\nu|^2 = |w|^2 + \frac{1}{n} \sum_{\nu=1}^{n} |z_\nu - w|^2,$$

where

$$w = \frac{1}{n} \sum_{\nu=1}^{n} z_\nu.$$

10.13 Let $z_1, z_2, \ldots, z_n$ be complex numbers such that

$$\alpha - \theta < \arg z_\nu < \alpha + \theta \quad (0 < \theta < \pi/2).$$

Prove the inequality

$$\left| \sum_{\nu=1}^{n} z_\nu \right| \geqq \cos\theta \sum_{\nu=1}^{n} |z_\nu|.$$

SOLUTION. We have

$$\left| \sum_{\nu=1}^{n} z_\nu \right| = \left| e^{-i\alpha} \sum_{\nu=1}^{n} z_\nu \right| \geqq \operatorname{Re}\left(e^{-i\alpha} \sum_{\nu=1}^{n} z_\nu \right)$$

$$= \sum_{\nu=1}^{n} |z_\nu| \cos(-\alpha + \arg z_\nu) \geqq \cos\theta \sum_{\nu=1}^{n} |z_\nu|.$$

Hence, the inequality is true.

§ 11. Miscellaneous Inequalities.

11.1 Prove that the following proposition is valid:

$$P(n)\colon f(n) = \sqrt{n+\sqrt{n-1+\sqrt{\ldots+\sqrt{2+\sqrt{1}}}}} < \sqrt{n}+1.$$

PROOF. $P(1)$ is valid, because certainly $1 < 2$. Suppose now that $P(n)$ is valid. Then,

$$\begin{aligned} f(n+1) &= \sqrt{n+1+f(n)} \\ &\leqq \sqrt{n+1+\sqrt{n}+1} \quad \text{(by the induction hypothesis)} \\ &\leqq \sqrt{(\sqrt{n}+1)^2} \\ &= \sqrt{n}+1 \\ &< \sqrt{n+1}+1. \end{aligned}$$

Thus, $P(n) \Rightarrow P(n+1)$.
This completes the inductive proof.

11.2 Prove the relation

$$(ca'-ac')^2 < 4(ab'-a'b)(bc'-b'c)$$
$$\Rightarrow (b^2-ac > 0 \text{ and } b'^2-a'c' > 0).$$

11.3 Determine the values of n for which the following inequality is valid

$$f_n(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2} \geqq \frac{n}{2}$$
$$(x_i \geqq 0;\ x_i+x_{i+1} > 0,\ i = 1, 2, \ldots, n;\ x_{n+1} = x_1). \tag{1}$$

PROOF for $n = 3$. For this case (1) reads

$$\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3} \geqq \frac{3}{2} \quad (x_2+x_3 = a_1,\ x_3+x_1 = a_2,\ x_1+x_2 = a_3). \tag{2}$$

We have

$$\left(\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3}\right)(x_1a_1+x_2a_2+x_3a_3)$$

$$= x_1^2+x_2^2+x_3^2+x_1x_2\left(\frac{a_2}{a_1}+\frac{a_1}{a_2}\right)+x_2x_3\left(\frac{a_3}{a_2}+\frac{a_2}{a_3}\right)+x_3x_1\left(\frac{a_1}{a_3}+\frac{a_3}{a_1}\right)$$

$$= (x_1+x_2+x_3)^2+\frac{x_1x_2}{a_1a_2}(a_1-a_2)^2+\frac{x_2x_3}{a_2a_3}(a_2-a_3)^2+\frac{x_3x_1}{a_3a_1}(a_3-a_1)^2.$$

Accordingly, it is sufficient to prove the inequality

$$(x_1+x_2+x_3)^2 \geqq \tfrac{3}{2}(x_1a_1+x_2a_2+x_3a_3) = 3(x_1x_2+x_2x_3+x_3x_1).$$

Since the latter may be written in the form

$$\tfrac{1}{2}\{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2\} \geqq 0,$$

we conclude that it is valid. Equality holds in (2) only if $x_1 = x_2 = x_3$.

The inequality (1) may be similarly proved for $n = 4, 5, 6$. (See **7.7** in reference to the case $n = 3$.)

REMARK. This problem was proposed by H. S. Shapiro (*The American Mathematical Monthly*, v. 61, 1954, p. 571). He proved (1) for $n = 3, 4$ (unpublished).

However, the proof of this inequality for $n = 3$ predates Shapiro's proposal. See, for example, G. L. Neviazhskii, *Inequalities*, Moscow 1947, pp. 130–131.

The challenge of the inequality (1) has created lively interest among mathematicians, as may be seen from the historical sketch which follows.

That (1) is not valid for all n was shown, by means of a counter-example, by Lighthill (*The Mathematical Gazette*, vol. 40, 1956, p. 266). Here is his counter-example:

Let $n = 20$, $x_{2k} = a_ke$, $x_{2k-1} = 1+b_ke$ $(k = 1, 2, \ldots, 10)$, where

$a_1 = 6$, $a_2 = 5$, $a_3 = 4$, $a_4 = 3$, $a_5 = 2$, $a_6 = 1$, $a_7 = 2$, $a_8 = 3$, $a_9 = 4$, $a_{10} = 5$,

$b_1 = 5$, $b_2 = 4$, $b_3 = 3$, $b_4 = 2$, $b_5 = 1$, $b_6 = 2$, $b_7 = 3$, $b_8 = 4$, $b_9 = 5$, $b_{10} = 6$.

and e is sufficiently small and positive.

We have

$$\begin{aligned}\frac{x_{2k-1}}{x_{2k}+x_{2k+1}} &= \frac{1+b_ke}{1+(a_k+b_{k+1})e}\\ &= (1+b_ke)\{1-(a_k+b_{k+1})e+(a_k+b_{k+1})^2e^2+O(e^3)\}\\ &= 1+(b_k-a_k-b_{k+1})e+(a_k+b_{k+1})(a_k+b_{k+1}-b_k)e^2+O(e^3),\end{aligned}$$

$$\frac{x_{2k}}{x_{2k+1}+x_{2k+2}} = \frac{a_k e}{1+(a_{k+1}+b_{k+1})e} = a_k e\{1-(a_{k+1}+b_{k+1})e+O(e^2)\}$$

$$= a_k e - a_k(a_{k+1}+b_{k+1})e^2+O(e^3) \quad (k = 1, 2, \ldots, 10),$$

where we have set

$$x_{21} = x_1,\ x_{22} = x_2,\ a_{11} = a_1,\ b_{11} = b_1.$$

Summing these twenty equalities we find

$$f_{20}(x_1, x_2, \ldots, x_{20}) = 10+pe+qe^2+O(e^3).$$

The coefficients p and q are:

$$p = \sum_{k=1}^{10} (b_k-a_k-b_{k+1})+\sum_{k=1}^{10} a_k = 0,$$

$$q = \sum_{k=1}^{10} (a_k+b_{k+1})(a_k+b_{k+1}-b_k)-\sum_{k=1}^{10} a_k(a_{k+1}+b_{k+1}) = -1,$$

and so the above relation becomes

$$f_{20}(x_1, x_2, \ldots, x_{20}) = 10-e^2+O(e^3).$$

Thus, for sufficiently small e, $f_{20}(x_1, x_2, \ldots, x_{20}) < 10$, which means that for $n = 20$ the inequality (1) is false.

Many writers have proved the truth of (1) for $n = 3, 4, 5, 6$ or for some of these values. A short proof of all these cases was given by L. J. Mordell (*Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, Band 22, Heft 3/4, 1958, p. 229–240). The proof given here for the case $n = 3$ was taken from this source. These results of Mordell suggest the conjecture that (1) is false for all $n \geqq 7$.

In addition to the aforementioned article by Mordell, there is the proof by A. Zulauf that (1) does not hold for even $n \geqq 14$. Zulauf proved that (1) fails for $n = 14$ by using Lighthill's method, and then applied induction.

This method of proof cannot be applied to the cases $n = 8, 10, 12$, because the coefficient q is then a positive-definite quadratic form in the variables a_k and b_k. On this point see: Dina Gladis S. Thomas (*The American Mathematical Monthly*, vol. 68, 1961, p. 472–473).

Here is Zulauf's proof: Analogous to the counter-example quoted above, we find that for $n = 14$ and sufficiently small e,

$$f_{14}(1+7e,\ 7e,\ 1+4e,\ 6e,\ 1+e,\ 5e,\ 1,\ 2e,\ 1+e,\ 0,\ 1+4e,\ e,\ 1+6e,\ 4e)$$
$$= 7-2e^2+O(e^3) < 7.$$

We suppose that (1) does not hold for $n = N$ and let $x_k (k = 1, 2, \ldots, N)$ be numbers such that

$$f_N(x_1, x_2, \ldots, x_N) < \tfrac{1}{2}N.$$

Then

$$f_{N+2}(x_1, x_2, \ldots, x_N, x_{N-1}, x_N) = f_N(x_1, x_2, \ldots, x_N)+1 < \tfrac{1}{2}(N+2),$$

i.e., (1) fails for $n = N+2$ also. Since (1) does not hold for $n = 14$, it follows that (1) fails to hold for all even $n \geqq 14$.

R. A. Rankin (An Inequality, *The Mathematical Gazette*, vol. 42, 1958, p. 39—40) proved the relation

$$\lim_{n\to\infty} \frac{\mu(n)}{n} = \inf_{n\geqq 3} \frac{\mu(n)}{n} < \tfrac{1}{2}-7\times 10^{-8} < \tfrac{1}{2},$$

where $\mu(n)$ is the lower bound of the function $f_n(x_1, x_2, \ldots, x_n)$ for $x_k \geqq 0$. Thus it follows that (1) does not hold for all sufficiently large n.

In a second article (*The Mathematical Gazette*, 1959, vol. 43, p. 182—184) A. Zulauf proved that (1) does not hold for $n = 53$, whence it follows that (1) is false for odd $n \geqq 53$.

D. Ž. Djoković has proved (*Proceedings of the Glasgow Mathematical Association*, vol. 6, part 1, 1963, p. 1—10) that (1) is true for $n = 8$. On the basis of this result P. H. Diananda (*Proceedings of the Glasgow Mathematical Association*, vol. 6, part 1, 1963, p. 11—13) and B. Bajšanski (*Publikacije Elektrotehničkog fakulteta Univerziteta u Beogradu*, serija: Matematika i fizika, No. 76, 1962) have proved independently of each other that (1) is also true for $n = 7$. In the same paper P. H. Diananda has proved that (1) does not hold for $n = 27$.

Hence, there is still undecided question whether (1) is true or not for $n = 9, 10, 11, 12, 13, 15, 17, 19, 21, 23$ and 25.

Equally interesting generalizations of this inequality are given by P. H. Diananda (*The American Mathematical Monthly*, vol. 66, 1959, p. 489—491).

11.4 Let

$$Q(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \ldots + \frac{x_{n-1}}{x_{n-1}+x_n} + \frac{x_n}{x_n+x_1},$$

where $x_1, x_2, \ldots, x_n \geqq 0$; $x_1+x_2, x_2+x_3, \ldots, x_{n-1}+x_n, x_n+x_1 > 0$.

Show that

$$1 < Q(x_1, x_2, \ldots, x_n) < n-1.$$

REMARK. This result was proved by A. Zulauf (*The Mathematical Gazette*, vol. 43, 1959, p. 42).

11.5 Determine whether the inequality

$$0 \leqq (x+a)^2/(x^2+x+1) \leqq \tfrac{4}{3}(a^2-a+1)$$

holds for all a and x.

11.6 If $x = 2t/(1+t^2)$, $y = (1-t^2)/(1+t^2)$, prove that

$$\frac{1}{2} \leqq \frac{7-6x-3y}{9-8x-3y} \leqq 1.$$

11.7 If $a_1, a_2, \ldots$ are distinct positive numbers, prove that

$$a_1^3+a_2^3 > a_1^2a_2+a_2^2a_1,$$
$$a_1^3+a_2^3+a_3^3 > a_1^2a_2+a_2^2a_3+a_3^2a_1,$$

and that in general, for $2 \leqq r$,

$$\sum_{k=1}^{r} a_k^3 > \left(\sum_{k=1}^{r} a_k^2 a_{k+1}\right) \quad (a_{r+1} = a_1).$$

11.8 Prove the inequality

$$\frac{p}{p+m} < \frac{p+q}{p+q+m+n} < \frac{q}{q+n}$$

$\left(\frac{m}{p} > \frac{n}{q}\text{; } m, n \text{ natural numbers; } p, q > 0\right)$.

Deduce that

$$\sum_{r=1}^{m} \frac{p}{p+r} + \sum_{s=1}^{n} \frac{q}{q+s} < \sum_{t=1}^{m+n} \frac{p+q}{p+q+t}.$$

In general, if $f(x)$ is a monotonic increasing function, show that

$$\sum_{r=1}^{m} f\left(\frac{p}{p+r}\right) + \sum_{s=1}^{n} f\left(\frac{q}{q+s}\right) < \sum_{t=1}^{m+n} f\left(\frac{p+q}{p+q+t}\right).$$

11.9 Prove that

$$\frac{x^n}{1+x+x^2+\ldots+x^{2n}} \leqq \frac{1}{2n+1} \quad (x > 0).$$

PROOF. Making use of

$$t+t^{-1} \geqq 2 \ (t = x^\nu, \nu = 1, 2, \ldots, n; x > 0)$$

we obtain

$$\frac{x^\nu}{\sum_{\nu=1}^{2n} x^n} = \frac{1}{1+\sum_{\nu=1}^{n} (x^\nu+x^{-\nu})} \leqq \frac{1}{2n+1}.$$

(Solution by R. Lučić.)

11.10 Prove the inequality

$$\frac{1+a+a^2+\ldots+a^n}{a+a^2+a^3+\ldots+a^{n-1}} \geqq \frac{n+1}{n-1} \quad (a>0;\, n \text{ a natural number} > 1). \tag{1}$$

PROOF. For $n = 2$, (1) is valid. Suppose that it holds for $n = k$, i.e., that

$$\frac{1+a+a^2+\ldots+a^k}{a+a^2+a^3+\ldots+a^{k-1}} \geqq \frac{k+1}{k-1}. \tag{2}$$

Since $a > 0$, (2) may be written in the form

$$1+a+\ldots+a^k \geqq \frac{k+1}{k-1}(a+a^2+\ldots+a^{k-1}), \tag{3}$$

whence we have

$$1+a+\ldots+a^k+a^{k+1} \geqq \frac{k+1}{k-1}(a+a^2+\ldots+a^{k-1})+a^{k+1}.$$

We shall establish

$$\frac{k+1}{k-1}(a+a^2+\ldots+a^{k-1})+a^{k+1} \geqq \frac{k+2}{k}(a+a^2+\ldots+a^{k-1}+a^k), \tag{4}$$

proving that (1) is also valid for $n = k+1$.

Let us assume that (4) does not hold, but instead that

$$\frac{k+1}{k-1}(a+a^2+\ldots+a^{k-1})+a^{k+1} < \frac{k+2}{k}(a+a^2+\ldots+a^{k-1}+a^k).$$

Then we find that

$$2(a+a^2+\ldots+a^{k-1}+a^k)+(k^2+k)a^k(a-1) < 0,$$

which is clearly impossible if $a \geqq 1$. Thus, by induction, (1) is established if $a \geqq 1$.

If we set $a = 1/b$ $(b > 0)$, the expression on the left-hand side of (1) becomes

$$f(b) = \frac{1+b+b^2+\ldots+b^n}{b+b^2+b^3+\ldots+b^{n-1}}.$$

The preceding proof shows that

$$f(b) \geqq (n+1)/(n-1) \text{ for } n > 1 \text{ and } b \geqq 1.$$

Thus, (1) is now established for all real a (> 0).

11.11 Show that if $a > b > 0$, then $A < B$, where:

$$A = \frac{1+a+\ldots+a^{n-1}}{1+a+\ldots+a^n}, \quad B = \frac{1+b+\ldots+b^{n-1}}{1+b+\ldots+b^n}.$$

PROOF.

$$\frac{1}{A} = 1+\frac{a^n}{1+a+\ldots+a^{n-1}}, \quad \frac{1}{B} = 1+\frac{b^n}{1+b+\ldots+b^{n-1}},$$

i.e.,

$$\frac{1}{A} = 1+\frac{1}{1/a^n+1/a^{n-1}+\ldots+1/a},$$

$$\frac{1}{B} = 1+\frac{1}{1/b^n+1/b^{n-1}+\ldots+1/b},$$

whence it follows that

$$1/A > 1/B, \text{ and so that } A < B.$$

11.12 Prove that

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \quad (n \text{ a natural number}).$$

PROOF. We apply the inequality

$$\frac{a_1+a_2+\ldots+a_n+a_{n+1}}{n+1} > \sqrt[n+1]{a_1a_2\ldots a_na_{n+1}} \quad (a_1, a_2, \ldots, a_{n+1} > 0)$$

with

$$a_1 = a_2 = \ldots = a_n = 1+1/n, \; a_{n+1} = 1$$

to obtain

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} < \frac{1+\left(1+\frac{1}{n}\right)n}{n+1} = 1+\frac{1}{n+1};$$

i.e.,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \quad (n \text{ a natural number}).$$

REMARK by D. C. B. Marsh. This result also follows from 2.30 by setting $b = n$, $a = n+1$.

11.13 Let $\{a_k\}$ $(k = 1, 2, \ldots, 2m)$ be a set of positive numbers and $\{a_k'\}$ $(k = 1, 2, \ldots, 2m)$ be this same set of numbers arranged in order of magnitude so that

$$a_1' \geqq a_2' \geqq a_3' \geqq \ldots \geqq a_{2m}'.$$

Prove L. A. Le Cointe's inequality

$$a_1'a_2' \ldots a_m' + a_{m+1}'a_{m+2}' \ldots a_{2m}' \geqq a_1a_2 \ldots a_m + a_{m+1}a_{m+2} \ldots a_{2m}.$$

REMARK. Concerning this generalized inequality see: T. Popoviciu, *Mathematica*, vol. 23, 1947—1948, p. 127—128.

11.14 Referred to a rectangular Cartesian coordinate system, consider the two points with coordinates:

$$\{\tfrac{1}{2}(v-u)-\tfrac{1}{2}\sqrt{3}(y-x),\ \tfrac{1}{2}\sqrt{3}(v-u)+\tfrac{1}{2}(y-x)\},\ \{w-u,\ z-x\},$$

where x, y, z, u, v, w are arbitrary real numbers. Calculate the square of the distance between these points and hence derive the inequality

$$(x^2+y^2+z^2)-(yz+zx+xy)+(u^2+v^2+w^2) - (vw+wu+uv) \geqq \sqrt{3}\begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}.$$

What conditions must x, y, z, u, v, w satisfy in order for equality to hold?

REMARK. See: H. Langman, *The American Mathematical Monthly*, vol. 35, 1928, p. 207.

For two proofs of this inequality (*loc. cit.*), see: F. Ayres, vol. 36, 1919, p. 238; and D. R. Curtiss, vol. 36, 1929, p. 289.

11.15 What conditions must be satisfied by the coefficients A, B, C, D, E, F for the function

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F \qquad (1)$$

to be positive for all real values of x and y?

ANSWER. The function (1) is positive for all x and y if and only if either

$$1°: \quad A > 0, \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0, \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} > 0,$$

or

$$2°: \quad A > 0, \begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0, \begin{vmatrix} A & B \\ D & E \end{vmatrix} = 0, \begin{vmatrix} A & D \\ D & F \end{vmatrix} > 0,$$

or

$$3°: \quad A = B = D = 0, \quad C > 0, \begin{vmatrix} C & E \\ E & F \end{vmatrix} > 0,$$

or

$$4°: \quad A = B = C = D = E = 0, \quad F > 0.$$

REMARK. The binomial $ax+b$ is positive for all x if $a = 0$ and $b > 0$. The trinomial $ax^2+2bx+c$ is positive for all x if either

$$a > 0, \; ac-b^2 > 0,$$

or

$$a = b = 0, \; c > 0.$$

GENERALIZATION. What conditions must be satisfied by the coefficients of the function

$$A_1x^2+A_2y^2+A_3z^2+2B_1yz+2B_2zx+2B_3xy \\ +2C_1x+2C_2y+2C_3z+D$$

if it is to assume only positive values for all values of the variables x, y, z?

11.16 Let $f(x) = a_0+a_1x+ \ldots +a_nx^n \, (a_i \geqq 0; i = 0, 1, 2, \ldots, n)$,
$$g(x) = f^2(x) = b_0+b_1x+ \ldots +b_{2n}x^{2n}.$$

Prove Moser's inequality:

$$b_{2r+1} \leqq \tfrac{1}{2}f^2(1).$$

PROOF. From the identity

$$\sum_{\nu=0}^{2n} b_\nu x^\nu = \left(\sum_{\nu=0}^{n} a_\nu x^\nu\right)^2, \tag{1}$$

by comparing the coefficients of x^{2r+1}, we find that

$$b_{2r+1} = \sum_{\nu=0}^{2r+1} a_\nu a_{2r+1-\nu} = 2\sum_{\nu=0}^{r} a_\nu a_{2r+1-\nu} \; (a_i = 0 \text{ for } i > n). \quad (2)$$

Note the relation

$$f(1) = \sum_{\nu=0}^{n} a_\nu = \sum_{\nu=0}^{r} (a_\nu + a_{2r+1-\nu}) + S_r, \quad (3)$$

where

$$S_r = \begin{cases} \sum_{\nu=2r+2}^{n} a_\nu & (n \geqq 2r+2), \\ 0 & (n < 2r+2). \end{cases}$$

On the basis of (3) and (2), we have in succession

$$f^2(1) \geqq \sum_{\nu=0}^{r} (a_\nu + a_{2r+1-\nu})^2$$

$$\geqq \sum_{\nu=0}^{r} (a_\nu + a_{2r+1-\nu})^2 - \sum_{\nu=0}^{r} (a_\nu - a_{2r+1-\nu})^2,$$

$$f^2(1) \geqq 4\sum_{\nu=0}^{r} a_\nu a_{2r+1-\nu},$$

$$\tfrac{1}{2}f^2(1) \geqq b_{2r+1}.$$

(Solution by D. Djoković.)

11.17 Prove the inequality

$$\sum_{k=0}^{n} \frac{a_k}{2^k} < 2,$$

where $\{a_k\}$ is the Fibonacci sequence defined by the relations

$$a_0 = 0,\ a_1 = 1,\ a_{k+2} = a_k + a_{k+1} \quad (k \text{ an integer} \geqq 0).$$

REMARK. Cf. P. S. Modenov: *Sbornik zadach po spetsial'nomu kursu elementarnoi matematiki*, Moscow 1957, p. 34, Problem 31.

11.18 If k is a natural number, prove that

$$\sum_{r=1}^{\infty} \frac{1}{(k+r)!} < \frac{k+2}{(k+1)!(k+1)}.$$

PROOF. Since

$$(k+2)(k+3)\ldots(k+r) > (k+2)^{r-1} \quad (r \geqq 2),$$

then

$$\sum_{r=1}^{\infty} \frac{1}{(k+r)!} < \frac{1}{(k+1)!} \sum_{r=1}^{\infty} \frac{1}{(k+2)^{r-1}} = \frac{1}{(k+1)!} \frac{k+2}{k+1},$$

which is the required result.

11.19 Prove that

$$\sum_{k=n+1}^{\infty} \frac{1}{k^a} < \frac{n^{1-a}}{a-1} \quad (n \text{ a natural number; } a > 1).$$

11.20 Prove that any real root x_1 of the equation

$$x^3+px+q = 0 \quad (p, q \text{ real numbers}) \tag{1}$$

satisfies the condition

$$p^2-4x_1 q \geqq 0. \tag{2}$$

Proof. Let x_1 be a real root of equation (1). Then x_1 is a root of the equation

$$x_1 x^2+px+q = 0. \tag{3}$$

Since equation (3) has real roots, its discriminant must be non-negative, i.e., the condition (2) must hold.

Generalization. Instead of equation (1), consider a more general equation, for example,

$x^m+px^n+q = 0$ (m, n natural numbers; p, q real numbers).

(Problem and proof by S. Prešić.)

11.21 For what values of x is the following inequality valid:

$$\sqrt{x+6} > \sqrt{x+1} + \sqrt{2x-5}?$$

Answer. The functions $\sqrt{x+6}$, $\sqrt{x+1}$, $\sqrt{2x-5}$ are all real if $x \geqq 5/2$.

The sense of the inequality remains unchanged if both sides are squared; thus,

$$x+6 > 3x-4+2\sqrt{2x^2-3x-5} \Rightarrow \sqrt{2x^2-3x-5} < 5-x.$$

The preceding inequality is meaningful when $5-x > 0$; i.e., $x < 5$.

After squaring, the preceding inequality becomes

$$(x-3)(x+10) < 0,$$

which requires x to satisfy, in addition to the earlier condition $x \in [5/2, 5)$, the condition $x \in (-10, 3)$.

CONCLUSION: The relation (1) is valid for $x \in [5/2, 3)$.

11.22 Solve the inequality

$$\sqrt{7x-13} - \sqrt{3x-19} > \sqrt{5x-27}.$$

ANSWER. $19/3 \leqq x < 9$.

11.23 Find the set of points (x, y) in the plane with coordinates which satisfy the simultaneous system of inequalities:

$$2x+3y-6 < 0,\ y+2 > 0,\ x+1 > 0,\ x^2+y^2-1 > 0.$$

11.24 Show that if $a \geqq 0$, $b \geqq 0$, $c \geqq 0$, $d \geqq 0$ and if $c+d \leqq \min(a, b)$, then

$$ad+bc \leqq ab,\ ac+bd \leqq ab.$$

PROOF 1. If

$$a \leqq b, \tag{1}$$

the inequality

$$c+d \leqq \min(a, b) \tag{2}$$

becomes

$$c+d \leqq a. \tag{3}$$

After multiplying the inequalities (1) and (3) by d and b, respectively, we obtain

$$(ad \leqq bd,\ bc+bd \leqq ab) \Rightarrow ad+bc \leqq ab.$$

If

$$b \leqq a, \tag{4}$$

the relation (2) becomes

$$c+d \leqq b. \tag{5}$$

From (4) and (5) we obtain

$$(bc \leqq ac,\ ac+ad \leqq ab) \Rightarrow ad+bc \leqq ab.$$

In an analogous manner, we may prove that $ac+bd \leqq ab$.

PROOF 2. Due to Underwood Dudley (*The American Mathematical Monthly*, vol. 65, 1958, p. 447):

$$\begin{aligned} ad+bc &\leqq (c+d)\max(a, b) \\ &\leqq \min(a, b)\max(a, b) \\ &= ab. \end{aligned}$$

REMARK. Can this result be generalized?

11.25 Prove the equivalence

$$|b-c| < a < b+c \Leftrightarrow |a-c| < b < a+c$$
$$(a, b, c \text{ real numbers}).$$

11.26 Prove that

$$\frac{e}{2x+2} < e-\left(1+\frac{1}{x}\right)^x < \frac{e}{2x+1} \quad (x > 0 \text{ or } x < -1). \quad (1)$$

PROOF. (1) may be written in the form

$$\frac{2x}{2x+1}\, e < \left(1+\frac{1}{x}\right)^x < \frac{2x+1}{2x+2}\, e. \quad (2)$$

1°: Let us suppose first that $x < -1$ and set $x = -y (y > 1)$. Then (2) becomes

$$\frac{1}{1-\dfrac{1}{2y}}\, e < \left(1-\frac{1}{y}\right)^{-y} < \frac{1-\dfrac{1}{2y}}{1-\dfrac{1}{y}}\, e. \quad (3)$$

If $A < B < C$ $(A, B, C > 0)$, then $\log A < \log B < \log C$. Applying this result to (3), we obtain the equivalent inequality

$$\begin{aligned} 1-\log\left(1-\frac{1}{2y}\right) &< -y\log\left(1-\frac{1}{y}\right) \\ &< 1+\log\left(1-\frac{1}{2y}\right) - \log\left(1-\frac{1}{y}\right). \quad (4) \end{aligned}$$

Since

$$\log(1-a) = -\sum_{k=1}^{\infty} \frac{a^k}{k} \quad (|a| < 1),$$

(4) takes the form

$$\sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{1}{2y}\right)^k < \sum_{k=2}^{\infty} \frac{1}{k+1}\left(\frac{1}{y}\right)^k < \sum_{k=2}^{\infty} \frac{1}{k}\left[\left(\frac{1}{y}\right)^k - \left(\frac{1}{2y}\right)^k\right] \quad (|y| > 1). \tag{5}$$

The coefficients of $(1/y)^k$ in the respective sums are

$$\frac{1}{k \cdot 2^k}, \ \frac{1}{k+1}, \ \frac{1}{k}\left(1-\frac{1}{2^k}\right).$$

We shall prove that

$$\frac{1}{k \cdot 2^k} < \frac{1}{k+1} < \frac{1}{k}\left(1-\frac{1}{2^k}\right) \quad (k = 2, 3, 4, \ldots). \tag{6}$$

Consider first the relation

$$\frac{1}{k \cdot 2^k} < \frac{1}{k+1}, \text{ i.e., } 2^k > \frac{k+1}{k}. \tag{7}$$

Since

$$2 > 1+\frac{1}{k} \quad (k > 1),$$

the inequality (7) is valid.

Let us now examine whether

$$\frac{1}{k+1} < \frac{1}{k}\left(1-\frac{1}{2^k}\right). \tag{8}$$

For $k = 2$, (8) is true. Let us suppose that it holds for some $k(\geqq 2)$, i.e., that $2^k > k+1$. After multiplying by 2, we obtain

$$2^{k+1} > 2k+2 > k+2 \quad (k \geqq 2).$$

The proof of (8) for $k \geqq 2$ may be completed by induction. Hence, the proof of the validity of (6) implies that of (2) and thence of (1) for all $x < -1$.

2°: Since

$$-1-z < -1 \text{ for } z > 0,$$

(2) is true, if we set $x = -1-z (z > 0)$. Thus

$$\frac{2z+2}{2z+1} e < \left(1 - \frac{1}{z+1}\right)^{-z-1} < \frac{2z+1}{2z} e \quad (z > 0).$$

After multiplication by $z/(z+1)$ (> 0), we obtain

$$\frac{2z}{2z+1} e < \left(1 + \frac{1}{z}\right)^{z} < \frac{2z+1}{2z+2} e \quad (z > 0).$$

This proves that (1) is also valid for $x > 0$.

11.27 Prove the inequality

$$\frac{x+y}{2} > \frac{x-y}{\log x - \log y} \quad (x > y > 0). \qquad (1)$$

SOLUTION. The inequality

$$\frac{\log x - \log y}{x-y} > \frac{2}{x+y} \quad (x > y > 0) \qquad (2)$$

is equivalent to (1). Putting $x = y+z$ $(z > 0)$ (2) becomes

$$\frac{\log(y+z) - \log y}{z} > \frac{2}{2y+z} \quad (y > 0; z > 0). \qquad (3)$$

Writing t instead of z/y (> 0) we obtain

$$\log\,(1+t) > \frac{2t}{t+2} = 2 - \frac{4}{t+2} \quad (t > 0). \qquad (4)$$

The function

$$f(t) = \log(1+t) + \frac{4}{t+2} - 2 \quad (t > -1)$$

is monotonically increasing since

$$f'(t) = \frac{1}{t+1} - \frac{4}{(t+2)^2} = \frac{t^2}{(t+1)(t+2)^2} > 0 \quad (t > -1).$$

Therefore, we have

$$f(t) > f(0) = 0 \quad (t > 0)$$

which is identical to (4).

REMARK. This solution is due to D. Djoković. Another solution can be found in the paper of B. Ostle and H. L. Terwilliger, *Proceedings of the Montana Academy of Sciences*, vol. 17, p. 69–70, 1957.

11.28 $3ab \leqq \sqrt[3]{a^2}+\sqrt[3]{b^2}+\frac{1}{2}(a^3+b^3) \quad (a, b \geqq 0).$

11.29 $\min\{(b-c)^2, (c-a)^2, (a-b)^2\} \leqq \frac{1}{2}(a^2+b^2+c^2).$

11.30 $\left|\frac{x^n-a^n}{x-a}-na^{n-1}\right| \leqq \frac{1}{2}n(n-1)r^{n-2}|x-a| \quad (|x| \leqq r, |a| \leqq r).$

11.31 $\sinh^{r+1} rx+\cosh^{r+1} rx < \cosh^r(r+1)x \quad (x > 0, r > 1).$

11.32 $\sinh^{r+1} rx+\cosh^{r+1} rx > \cosh^r(r+1)x \quad (x > 0, 0 < r < 1).$

11.33 $\sqrt{a_1a_n} \leqq \sqrt[n]{a_1a_2\ldots a_n} \leqq \frac{1}{2}(a_1+a_n)$
$\{a_k = a_1+(k-1)d;\ a_1, d \geqq 0\}.$

11.34 $\prod_{k=1}^{n}\left(\frac{s}{a_k}-1\right)^{a_k} \leqq (n-1)^s \quad \left(a_k > 0, s = \sum_{k=1}^{n} a_k\right).$

11.35 $\left|\sqrt{a_1{}^2+a_2{}^2+\ldots+a_n{}^2}-\sqrt{b_1{}^2+b_2{}^2+\ldots+b_n{}^2}\right|$
$\leqq |a_1-b_1|+|a_2-b_2|+\ldots+|a_n-b_n|.$

11.36 $\frac{\pi}{2} < \sum_{n=1}^{\infty}\frac{1}{(n+1)\sqrt{n}} < \frac{\pi+1}{2}.$

11.37 $\frac{\pi}{2}-\frac{1}{a} < \sum_{n=1}^{\infty}\frac{a}{a^2+n^2} < \frac{1}{2}\pi \quad (a > 0).$

11.38 $\sum_{k=n+1}^{\infty}\frac{1}{k^2+ak+1} < \frac{\pi}{\sqrt{4-a^2}}-\frac{2}{\sqrt{4-a^2}}\arctan\frac{2n+a+2}{\sqrt{4-a^2}}$
$+\frac{1}{n^2+an+1} \quad (0 < a < 2).$

11.39 $\sum_{k=n+1}^{\infty}\frac{1}{k^2+2k+1} < \frac{1}{n+2}+\frac{1}{(n+1)^2}.$

11.40 $\sum_{k=n+1}^{\infty}\frac{1}{k^2+ak+1} < \frac{1}{\sqrt{a^2-4}}\log\frac{2n+2+a+\sqrt{a^2-4}}{2n+2+a-\sqrt{a^2-4}}$
$+\frac{1}{n^2+an+1} \quad (a > 2).$

11.41 $\sum_{n=1}^{\infty}\frac{1}{n^3} < \frac{5}{4}.$